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Pseudo-differential type Operators and Their Multiplication involving Hankel type Convolution

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Abstract

In the present paper we have defined pseudo-differential type operators $A(x, D), B(y, D)$ in terms of two symbols. Further the multiplication of these two operators aldo defined. It is also shown that the pseudo-differential type opeartor and multiplication of pseudo-differential type operators are bounded in certain Sobolev type space associated with the Hankel type transform. Finally some special cases are studied.

1 Introduction

The Hankel type transform of $\phi \in L^1(I), I = (0, \infty)$ is defined by

$$(H_{\alpha,\beta}\phi)(x) = \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\phi(y)y^{4\alpha}dy, x \in I \tag{1.1}$$

where $J_{\alpha-\beta}$ denotes the Bessel type function of the first kind and order $(\alpha-\beta)$. Througtout this paper we assume that $(\alpha - \beta) \geq -1/2$. We note that if ϕ is a Lebegue measurable function on I and

$$\int_0^\infty x^{4\alpha}|\phi(x)|dx < \infty, \tag{1.2}$$

then as the function $t^{-(\alpha-\beta)}J_{\alpha-\beta}(t)$ is bounded on I, the Hankel type transformation $H_{\alpha,\beta}(\phi)$ is bounded on I. The inverse formula for (1.1) is given by

$$\phi(y) = \int_0^\infty \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\,(H_{\alpha,\beta}\phi)(x)x^{4\alpha}dx, y \in I \tag{1.3}$$

Following Zemanian [10], we introduce the space $H^{\alpha,\beta}$ as the space of all those complex valued and smooth functions ϕ defined on I such that for every $m, n \in \mathbb{N}_0$

$$\rho_{m,n}^{\alpha,\beta}(\phi) = Sup_{x\in I} \left(1 + x^2\right)^m \left|\left(x^{-1}D\right)^n x^{2\beta-1}\phi(x)\right| < \infty \tag{1.4}$$

On $H^{\alpha,\beta}$, the topology generated by the family $\left\{\rho_{m,n}^{\alpha,\beta}\right\}_{m,n\in\mathbb{N}_0}$ of seminorms. Then $H^{\alpha,\beta}$ is a Frechet space and Hankel type transformation $h_{\alpha,\beta}$ defined by

$$\left(h_{\alpha,\beta}\phi\right)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\phi(y)dy, x \in I \tag{1.5}$$

We have following lemma:

Lemma 1 *The two forms $H_{\alpha,\beta}$ and $h_{\alpha,\beta}$ of Hankel type transforms are related through*

$$\left(H_{\alpha,\beta}\phi\right)(x) = x^{2\beta-1} h_{\alpha,\beta}\left(y^{2\alpha}\phi\right)(x), x \in I \tag{1.6}$$

Proof 1 *By (1.1), we have*

$$\begin{aligned}
\left(H_{\alpha,\beta}\phi\right)(x) &= \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\phi(y)y^{4\alpha}dy, x \in I \\
&= \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\phi(y)x^{2\beta-1}y^{2\alpha}dy \\
&= x^{2\beta-1}\int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\phi(y)\left(y^{2\alpha}\phi\right)(y)dy \\
&= x^{2\beta-1} h_{\alpha,\beta}\left(y^{2\alpha}\phi\right)(x)
\end{aligned}$$

Thus lemma is proved.

Now for $1 \leq p < \infty$, we define the space $L_{\sigma,p}$ as the space of all those measueable functions ϕ on I such that

$$||\phi||_{L_{\sigma,p}} = \left[\int_0^\infty |\phi(x)|^p d\sigma(x)\right]^{1/p} < \infty \tag{1.7}$$

By $L_{\sigma,\infty}$ we represent the space of essentially (with respect to the measure $x^{4\alpha}dx$ or equivalently with respect to Lebesgue measure) bounded functions on I. The usual norm in $L_{\sigma,\infty}$ is denoted by $||\ \ ||_{\sigma,\infty}$
If $f \in L_{\sigma,p}$ for some $1 \leq p < \infty$ then f defines an element of $H^{\alpha,\beta}$ through

$$\langle f, \phi\rangle = \int_0^\infty f(x)\phi(x)x^{4\alpha}dx, \phi \in H^{\alpha,\beta} \tag{1.8}$$

Following Hirschman [5] and haimo [4], we define the hankel type convolution on $L_{\sigma,p}$ by

$$(f\#g)(x) = \int_0^\infty f(y)\left(\tau_x g\right)(y)d\sigma(y), \tag{1.9}$$

where the Hankel type translation operator $\tau_x, x \in I$ is defined through

$$\left(\tau_x g\right)(y) = \int_0^\infty g(z) D_{\alpha,\beta}(x,y,z) d\sigma(z) \tag{1.10}$$

provided the above intgral exists, where

$$D_{\alpha,\beta}(x,y,z) = \left[2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right]^2 \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt)(yt)^{-(\alpha-\beta)} J_{\alpha-\beta}(yt) (zt)^{-(\alpha-\beta)} J_{\alpha-\beta}(zt) t^{4\alpha} dt \tag{1.11}$$

$$j_{\alpha-\beta}(x) = 2^{\alpha-\beta}\Gamma(3\alpha+\beta) x^{-(\alpha-\beta)} J_{\alpha-\beta}(x) \tag{1.12}$$

$$d\sigma(x) = \frac{x^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)} dx \tag{1.13}$$

The following interchange formula holds,

$$H_{\alpha,\beta}\left(f \# g\right) = H_{\alpha,\beta}(f) H_{\alpha,\beta}(g) \tag{1.14}$$

and

$$\left(f\#g\right)\#h(x) = f\#\left(g\#h\right), f,g,h \in L_{\sigma,p}$$

If $f \in L_{\sigma,1}, g \in L_{\sigma,p}$ then the integral defining $f\#g(x)$ converges for all x and

$$||f\#g||_{L_{\sigma,p}} \leq ||f||_{L_{\sigma,1}} ||g||_{L_{\sigma,p}} \tag{1.15}$$

From [9] we have,

$$\left(H_{\alpha,\beta,a}\phi\right)(x) = \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy) a(x,y) y^{4\alpha} \left(H_{\alpha,\beta}\phi\right)(y) dy \tag{1.16}$$

where

$$\left(H_{\alpha,\beta}\phi\right)(x) = \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\phi(y) y^{4\alpha} dy, (\alpha-\beta) \geq -1/2 \tag{1.17}$$

According to Rodino [6] the symbol $a(x,y)$ is defined to be the complex valued infinitely differentiable function on $I \times I$ which satisfies

$$\left|\left(x^{-1}D_x\right)^a \left(y^{-1}D_y\right)^b a(x,y)\right| \leq C^{a+b+1} a!b!(1+y)^{m-\rho b+\delta a} \tag{1.18}$$

for all $a,b \in \mathbb{N}_0$, where C is a constant and ρ,δ are real numbers such that $0 \leq \delta < \rho < 1$ and m is a fixed real number. The class of such symbols are denoted by $H(m)$.
Following [1], we define the the Bessel type operator as:

$$\Delta_{\alpha,\beta} = x^{4\beta-2} D x^{4\alpha} D \tag{1.19}$$

The Bessel type operator $\Delta_{\alpha,\beta}$ can also be written as

$$\Delta_{\alpha,\beta} = 4\alpha x^{4(\alpha+\beta)-3} D_x + x^{4(\alpha+\beta)-2} D_x^2 \tag{1.20}$$

From [1], for $1 \leq p < \infty$, we say that a measurable function $f \in I$ is in $H_{\alpha,\beta,p}$ if for every $k \in \mathbb{N}_0, \Delta_{\alpha,\beta}^k f \in L_{\alpha,\beta,p}$ such that

$$\left\langle \Delta_{\alpha,\beta}^k f, \phi \right\rangle = \int_0^\infty \left(\Delta_{\alpha,\beta}^k \phi \right)(x) f(x) x^{4\alpha} dx, \phi \in H^{\alpha,\beta} \tag{1.21}$$

and if $f \in H_{\alpha,\beta,p}$ with $1 \leq p \leq 2$ then from [[11] lemma 5.4] we have

$$H_{\alpha,\beta}\left(\Delta_{\alpha,\beta}^k f\right) = \left(-y^2\right)^k H_{\alpha,\beta}(f) \tag{1.22}$$

and from [9] we have

$$\left(x^{-1} D_x\right)^k (\phi\psi) = \sum_{r=0}^{k} kCr \left(x^{-1} D_x\right)^r \phi \left(x^{-1} D_x\right)^{k-r} \psi \tag{1.23}$$

We require following definition:

Definition 1 (Sobolev type space) *The space $G_{\alpha,\beta,p}^s(I), s \in \mathbb{R}, (\alpha-\beta) \in \mathbb{R}$ isn defined to be the set of all those elements $\phi \in H_1^{\alpha,\beta}$ which satisfy*

$$||\phi||_{G_{\alpha,\beta,p}^s} = \left|\left|(1+\eta^2)^s H_{\alpha,\beta}\phi\right|\right|_{L_{\sigma,p}}, 1 \leq p \leq \infty \tag{1.24}$$

2 Pseudo-differential type operator $A(x, D)$

Definition 2 *We define the pseudo-differential type operator $A(x, D)$ as*

$$A(x,D)\phi(x) = \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) a(x,t) \left(H_{\alpha,\beta}\phi\right)(t) t^{4\alpha} dt \tag{2.1}$$

where $\phi \in H^{\alpha,\beta}(I), I = (0,\infty), (\alpha-\beta) \geq -1/2$ and

$$a(x,t) = \int_0^\infty (x\lambda)^{(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) b(\lambda,t) \lambda^{4\alpha} d\lambda \tag{2.2}$$

with the condition that for all $\lambda \in I, t \in I$,

$$|b(\lambda,t)| \leq k(\lambda) \in L_{\sigma,1}(I) \tag{2.3}$$

Theorem 1 *Let $(\alpha-\beta) \geq -1/2$ then*

$$||A(x,D)\phi(x)||_{G_{\alpha,\beta,1}^0} \leq ||k||_{L_{\sigma,1}} ||\phi||_{G_{\alpha,\beta,1}^0}, \phi \in H^{\alpha,\beta}(I)$$

Proof 2 *From (2.1) and (2.2), we have*

$$A(x,D)\phi(x) = \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \left(\int_0^\infty (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) b(\lambda,t) \lambda^{4\alpha} d\lambda \right) \times (H_{\alpha,\beta}\phi)(t) t^{4\alpha} dt$$

Therefore

$$\begin{aligned}
\left[H_{\alpha,\beta}\left(A(x,D)\right)\phi(x)\right](z) &= \int_0^\infty (zx)^{-(\alpha-\beta)} J_{\alpha-\beta}(zx) A(x,D)\phi(x) x^{4\alpha} dx \\
&= \int_0^\infty \int_0^\infty \int_0^\infty (zx)^{-(\alpha-\beta)} J_{\alpha-\beta}(zx)(xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \\
&\times (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) b(\lambda,t) \left(H_{\alpha,\beta}\phi\right)(t)(x\lambda)^{4\alpha} dt d\lambda dx \\
&= \int_0^\infty \int_0^\infty b(\lambda,t)\lambda^{4\alpha} \left(H_{\alpha,\beta}\phi\right)(t) t^{4\alpha} \int_0^\infty (zx)^{-(\alpha-\beta)} J_{\alpha-\beta}(zx) \\
&\times (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt)(x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) x^{4\alpha} dx dt d\lambda
\end{aligned}$$

Now using inequality (1.11), the last expression can be written as

$$\begin{aligned}
\left[H_{\alpha,\beta}\left(A(x,D)\right)\phi(x)\right](z) &= \frac{1}{\left[2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right]^2} \int_0^\infty \int_0^\infty b(\lambda,t)\lambda^{4\alpha} \left(H_{\alpha,\beta}\phi\right)(t) t^{4\alpha} \\
&\times D_{\alpha,\beta}(t,\lambda,z) dt d\lambda \\
&= \int_0^\infty \int_0^\infty b(\lambda,t) \left(H_{\alpha,\beta}\phi\right)(t) D_{\alpha,\beta}(t,\lambda,z) d\sigma(t) d\sigma(\lambda).
\end{aligned}$$

Now using inequality (2.3), we obtain

$$\begin{aligned}
\left|\left[H_{\alpha,\beta}\left(A(x,D)\right)\phi(x)\right](z)\right| &\le k(\lambda) \left(\int_0^\infty \left|\left(H_{\alpha,\beta}\phi\right)(t)\right| D_{\alpha,\beta}(t,\lambda,z) d\sigma(t) d\sigma(\lambda) \right) \\
&\le k(\lambda) \left(\tau_z \left(H_{\alpha,\beta}\phi\right)\right)(\lambda) d\sigma(\lambda)
\end{aligned}$$

In view of [11] and [4], we have

$$\left|\left[H_{\alpha,\beta}\left(A(x,D)\right)\phi(x)\right](z)\right| \le \left(k \# \left|H_{\alpha,\beta}\phi\right|\right)(z).$$

Thus

$$\int_0^\infty \left|\left[H_{\alpha,\beta}\left(A(x,D)\right)\phi(x)\right](z)\right| d\sigma(z) \le \int_0^\infty \left(k \# \left|\left(H_{\alpha,\beta}\phi\right)(t)\right|\right)(z) d\sigma(z).$$

or

$$\left|\left|H_{\alpha,\beta}\left(A(x,D)\right)\phi(x)\right|\right|_{L_{\sigma,1}} \le \left|\left|k \# \left|\left(H_{\alpha,\beta}\phi\right)(t)\right|\right|\right|_{L_{\sigma,1}}$$

Using the inequality (1.15), we have

$$\left|\left|H_{\alpha,\beta}\left(A(x,D)\right)\phi(x)\right|\right|_{L_{\sigma,1}} \le ||k||_{L_{\sigma,1}} \left|\left|H_{\alpha,\beta}\phi\right|\right|_{L_{\sigma,1}}$$

Now applying definition (1.1) we obtain

$$||A(x,D)\phi(x)||_{G^0_{\alpha,\beta,1}} \le ||k||_{L_{\sigma,1}} ||\phi||_{G^0_{\alpha,\beta,1}}, \phi \in H^{\alpha,\beta}(I)$$

The proof is completed.

3 Pseudo-differential type operator $B(y, D)$

Definition 3 *We define the pseudo-differential type operator $B(y, D)$ as*

$$B(y, D)\phi(y) = \int_0^\infty (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) e(y, s) \left(H_{\alpha,\beta}\phi\right)(s) s^{4\alpha} ds, \phi \in H^{\alpha,\beta}(I) \tag{3.1}$$

where

$$e(y, s) = \int_0^\infty (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) f(b, s) b^{4\alpha} db \tag{3.2}$$

with the condition that for all $b, s \in I$ and $(\alpha - \beta) \geq -1/2$,

$$|f(b, s)| \leq l(b) \in L_{\sigma,1}(I) \tag{3.3}$$

Theorem 2 *Let $(\alpha - \beta) \geq -1/2$, then*

$$||B(y, D)\phi||_{G^0_{\alpha,\beta,1}} \leq ||k||_{L_{\sigma,1}} ||\phi||_{G^0_{\sigma,1}}, \phi \in H^{\alpha,\beta}(I)$$

Proof 3 *Proceeding as in the proof of theorem 1 we find that*

$$\begin{aligned}
\left[H_{\alpha,\beta}\left(B(y, D)\right)\phi(y)\right](z) &= \left|\int_0^\infty (zy)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) \left[B(y, D)\phi(y)\right] y^{4\alpha} dy\right| \\
&= \int_0^\infty (zy)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) \int_0^\infty (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \\
&\times e(y, s) \left(H_{\alpha,\beta}\phi\right)(s) s^{4\alpha} ds y^{4\alpha} dy \\
&= \int_0^\infty \int_0^\infty (zy)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \\
&\times \int_0^\infty (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) f(b, s) b^{4\alpha} db \left(H_{\alpha,\beta}\phi\right)(s) s^{4\alpha} ds y^{4\alpha} dy \\
&= \int_0^\infty \int_0^\infty \int_0^\infty (zy)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \\
&\times (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) f(b, s) (ybs)^{4\alpha} \left(H_{\alpha,\beta}\phi\right)(s) db ds dy
\end{aligned}$$

Now by using (1.7),(1.10),(1.11) and (3.3) we get

$$\left|\left[H_{\alpha,\beta}\left(B(y, D)\right)\phi(y)\right](z)\right| \leq \left(l \# \left|H_{\alpha,\beta}\phi\right|\right)(z) \tag{3.4}$$

From which the assertion follows. Thus proof is completed.

4 Multiplication of Pseudo-differential type operators

Definition 4 *The multiplication of two pseudo-differential type operators $A(x,D)$ and $B(y,D)$ associated with symbols $a(x,t)$ and $b(y,s)$ respectively is defined by*

$$A(x,D)B(y,D)\phi(x)=\int_0^\infty (xt)^{-(\alpha-\beta)}J_{\alpha-\beta}(xt)a(x,t)H_{\alpha,\beta}\left(B(y,D)\phi\right)(t)t^{4\alpha}dt \tag{4.1}$$

From (2.2),(3.1) and (3.2), we have

$$\begin{aligned}A(x,D)B(y,D) &= \int_0^\infty (xt)^{-(\alpha-\beta)}J_{\alpha-\beta}(xt)\int_0^\infty (x\lambda)^{(\alpha-\beta)}J_{\alpha-\beta}(x\lambda)b(\lambda,t)\lambda^{4\alpha}d\lambda \\ &\times\int_0^\infty (ty)^{-(\alpha-\beta)}J_{\alpha-\beta}(ty)B(y,D)\phi(y)y^{4\alpha}dyt^{4\alpha}dt \\ &= \int_0^\infty\int_0^\infty\int_0^\infty (xt)^{-(\alpha-\beta)}J_{\alpha-\beta}(xt)t^{4\alpha}(x\lambda)^{(\alpha-\beta)}J_{\alpha-\beta}(x\lambda)b(\lambda,t)\lambda^{4\alpha} \\ &\times (ty)^{-(\alpha-\beta)}J_{\alpha-\beta}(ty)\int_0^\infty (ys)^{-(\alpha-\beta)}J_{\alpha-\beta}(ys)e(y,s)\left(H_{\alpha,\beta}\phi\right)(s) \\ &\times s^{4\alpha}ds\phi(y)y^{4\alpha}dyd\lambda dt \\ &= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (xt)^{-(\alpha-\beta)}J_{\alpha-\beta}(xt)(x\lambda)^{(\alpha-\beta)}J_{\alpha-\beta}(x\lambda) \\ &\times b(\lambda,t)(ty)^{-(\alpha-\beta)}J_{\alpha-\beta}(ty)(ys)^{-(\alpha-\beta)}J_{\alpha-\beta}(ys) \\ &\times\int_0^\infty (sb)^{-(\alpha-\beta)}J_{\alpha-\beta}(sb)f(b,s)b^{4\alpha}db\left(H_{\alpha,\beta}\phi\right)(s) \\ &\times (y\lambda ts)^{4\alpha}\phi(y)dyd\lambda dtds \\ &= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (xt)^{-(\alpha-\beta)}J_{\alpha-\beta}(xt)(x\lambda)^{(\alpha-\beta)}J_{\alpha-\beta}(x\lambda) \\ &\times (ty)^{-(\alpha-\beta)}J_{\alpha-\beta}(ty)(ys)^{-(\alpha-\beta)}J_{\alpha-\beta}(ys)(sb)^{-(\alpha-\beta)}J_{\alpha-\beta}(sb) \\ &\times b(\lambda,t)f(b,s)\left(H_{\alpha,\beta}\phi\right)(s)(y\lambda tsb)^{4\alpha}\phi(y)dyd\lambda dtdsdb\end{aligned}$$

provided multiple integral exists.

Theorem 3 *Let $(\alpha-\beta)\geq -1/2$ then*

$$||A(x,D)B(x,D)\phi(x)||_{G^0_{\alpha,\beta,1}}\leq ||k||_{L_{\sigma,1}}||\phi||_{G^0_{\alpha,\beta,1}} \tag{4.2}$$

Proof 4 *From definition (4.1) and (2.2), we have*

$$\begin{aligned} A(x,D)B(x,D)\phi(x) &= \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) a(x,t) H_{\alpha,\beta}\left(B(y,D)\phi\right)(t) t^{4\alpha} dt \\ &= \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \int_0^\infty (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) \\ &\times b(\lambda,t)\lambda^{4\alpha} d\lambda H_{\alpha,\beta}\left(B(y,D)\phi\right)(t) t^{4\alpha} dt \end{aligned}$$

Therefore

$$\begin{aligned} &H_{\alpha,\beta}\left(A(x,D)B(x,D)\phi(x)\right)(z) \\ &= \int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) \left(A(x,D)B(y,D)\phi(x)\right) \\ &\times x^{4\alpha} dx \\ &= \int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) \int_0^\infty (xt)^{-(\alpha-\beta)} \\ &\times J_{\alpha-\beta}(xt) a(x,t) H_{\alpha,\beta}\left(B(y,D)\phi(x)\right) t^{4\alpha} dt x^{4\alpha} dx \\ &= \int_0^\infty \int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz)(xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \\ &\times \int_0^\infty (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) b(\lambda,t)\lambda^{4\alpha} d\lambda H_{\alpha,\beta} \\ &\times \left(B(y,D)\phi(x)\right)(xt)^{4\alpha} dt dx \\ &= \int_0^\infty \int_0^\infty \int_0^\infty b(\lambda,t) H_{\alpha,\beta}\left(B(y,D)\phi(x)\right) \\ &\times (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz)(xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \\ &\times (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda)(xt\lambda)^{4\alpha} d\lambda dt dx \end{aligned}$$

Now using (1.11) and (1.13), we have

$$\begin{aligned} &\le \frac{1}{\left(2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right)^2} \int_0^\infty \int_0^\infty b(\lambda,t) H_{\alpha,\beta}\left(B(y,D)\phi(x)\right) D_{\alpha,\beta}(z,t,\lambda)(\lambda t)^{4\alpha} dt d\lambda \\ &\le \int_0^\infty \int_0^\infty b(\lambda,t) H_{\alpha,\beta}\left(B(y,D)\phi(x)\right) D_{\alpha,\beta}(z,t,\lambda)(\lambda t)^{4\alpha} d\sigma(t) d\sigma(\lambda) \end{aligned}$$

By using inequalities (2.3) and (3.4) we have

$$H_{\alpha,\beta}\left(A(x,D)B(x,D)\phi(x)\right)(z) \le \int_0^\infty \int_0^\infty k(\lambda)\left(l\#\left|H_{\alpha,\beta}\phi\right|\right)(t) D_{\alpha,\beta}(z,t,\lambda) d\sigma(t) d\sigma(\lambda)$$

Now by applying equations (1.11) and (1.10) we have

$$H_{\alpha,\beta}\left(A(x,D)B(x,D)\phi(x)\right)(z) \leq \int_0^\infty k(\lambda)\tau_z\left(l\#\left|H_{\alpha,\beta}\phi\right|\right)(\lambda)d\sigma(\lambda) \tag{4.3}$$
$$\leq k\#\left(l\#\left|H_{\alpha,\beta}\phi\right|\right)(z) \tag{4.4}$$

Hence

$$\int_0^\infty H_{\alpha,\beta}\left(A(x,D)B(x,D)\phi(x)\right)(z)d\sigma(z) \leq \int_0^\infty k\#\left(l\#\left|H_{\alpha,\beta}\phi\right|\right)(z)d\sigma(z)$$
$$\leq ||k||_{L_{\sigma,1}}\,||l\#\left|H_{\alpha,\beta}\phi\right|||_{L_{\sigma,1}}$$

So that

$$||H_{\alpha,\beta}\left(A(x,D)B(x,D)\right)||_{L_{\sigma,1}} \leq ||k||_{L_{\sigma,1}}\,||l||_{L_{\sigma,1}}\,||H_{\alpha,\beta}\phi||_{L_{\sigma,1}}$$

Using theorem 1 we have

$$||(A(x,D)B(x,D)||_{G^0_{\alpha,\beta,1}} \leq ||l||_{L_{\sigma,1}}\,||\phi||_{G^0_{\alpha,\beta,1}}$$

Thus proof is completed.

5 Some Special Cases

In this section, we study some special cases as following theorems

Theorem 4 *(i) Let*

$$b(\lambda,t) = b_1(\lambda)b_2(t) \tag{5.1}$$

then

$$||A(x,D)\phi(x)||_{G^0_{\alpha,\beta,1}} \leq ||b_1||_{L_{\sigma,1}}\,||b_2H_{\alpha,\beta}\phi||_{L_{\sigma,1}}$$

and

$$A(x,D)\phi(x) = \left(H_{\alpha,\beta}b_1\right)(x)\left(H_{\alpha,\beta}b_2H_{\alpha,\beta}\phi\right)(x)$$

(ii) Let

$$f(b,s) = f_1(b)f_2(s) \tag{5.2}$$

then

$$||B(y,D)\phi(y)||_{G^0_{\alpha,\beta,1}} \leq ||f_1||_{L_{\sigma,1}}\,||f_2H_{\alpha,\beta}\phi||_{L_{\sigma,1}}$$

and

$$B(y,D)\phi(x) = \left(H_{\alpha,\beta}f_1\right)(y)\left(H_{\alpha,\beta}f_2H_{\alpha,\beta}\phi\right)(y)$$

Proof 5 *here we prove (ii). (i) can be proved similarly. From (3.1), (3.2) and (5.2), we have*

$$B(y,D)\phi(y) = \int_0^\infty (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \int_0^\infty (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) f_1(b) f_2(s) b^{4\alpha} db \tag{5.3}$$

$$\times H_{\alpha,\beta}\phi(s) s^{4\alpha} ds \tag{5.4}$$

$$= \int_0^\infty \int_0^\infty (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys)(yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) f_1(b) f_2(s) \tag{5.5}$$

$$\times H_{\alpha,\beta}\phi(s)(bs)^{4\alpha} dbds \tag{5.6}$$

From equation (1.11)

$$D_{\alpha,\beta}(s,b,z) = \left[2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right]^2 \int_0^\infty (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \int_0^\infty (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) \times (zy)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) y^{4\alpha} dy$$

$$\int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) D_{\alpha,\beta}(s,b,z) z^{4\alpha} dz = \left[2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right]^2 (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \tag{5.7}$$

$$\times \int_0^\infty (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) \tag{5.8}$$

Thus from above, we have

$$\begin{aligned} B(y,D)\phi(y) &= \frac{1}{\left[2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right]^2} \int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) \int_0^\infty \int_0^\infty f_1(b) f_2(s) \\ &\times H_{\alpha,\beta}\phi(s)(bs)^{4\alpha} D_{\alpha,\beta}(s,b,z) z^{4\alpha} dzdbds \\ &= \frac{1}{\left[2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right]^2} \int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) \int_0^\infty f_1(b) \\ &\times \int_0^\infty \left(f_2 H_{\alpha,\beta}\phi\right)(s) D_{\alpha,\beta}(s,b,z) d\sigma(s)(bz)^{4\alpha} dzdb \\ &= \int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) \int_0^\infty f_1(b)\tau_z\left(f_2(H_{\alpha,\beta}\phi)(b)d\sigma(b)\right) z^{4\alpha} dz \\ &= \int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz)\left(f_1 \# f_2(H_{\alpha,\beta}\phi)\right)(z) z^{4\alpha} dz \end{aligned}$$

An application of the inverse Hankel type transform yields

$$\int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) B(y,D)\phi(y) y^{4\alpha} dy = \left(f_1 \# f_2(H_{\alpha,\beta}\phi)\right)(z)$$

or in other words,

$$H_{\alpha,\beta}\left[B(y,D)\phi(y)\right](z) = \left(f_1 \# f_2(H_{\alpha,\beta}\phi)\right)(z) \tag{5.9}$$

Thus we have

$$\int_0^\infty H_{\alpha,\beta}\left[B(y,D)\phi(y)\right](z)d\sigma(z) = \int_0^\infty \left(f_1\#f_2(H_{\alpha,\beta}\phi)\right)(z)d\sigma(z)$$

Now by applying definition(1), inequality (1.15) we get

$$||H_{\alpha,\beta}\left[B(y,D)\phi(y)\right]||_{L_{\sigma,1}} = ||(f_1\#f_2(H_{\alpha,\beta}\phi))||_{L_{\sigma,1}}$$
$$||B(y,D)\phi(y)||_{G^0_{\alpha,\beta,1}} = ||f_2 H_{\alpha,\beta}\phi||_{L_{\sigma,1}}$$

Now from (5.6), using Hankel type inversion formula, we obtain

$$B(y,D)\phi(y) = H_{\alpha,\beta}\left(f_1\#f_2 H_{\alpha,\beta}\phi\right)(y)$$

By using (1.14) we get $B(y,D)\phi(y) = \left(H_{\alpha,\beta}f_1\right)(y)\left(H_{\alpha,\beta}f_2H_{\alpha,\beta}\phi\right)(y)$
Thus the proof is completed.

Theorem 5 *(i) Let* $b(\lambda,t) = b_1(\lambda)b_2(t)$ *and* $f(b,s) = f_1(b)f_2(s)$*, where* $b_2 = A$ *and* $f_2 = B$ *are assumed to be constants. Assume further that* $b_1(\lambda) \in L_{\sigma,1}(I)$ *and* $f_1(b)L_{\sigma,1}$*. Then*

$$||A(x,D)B(x,D)\phi(x)||_{G^0_{\alpha,\beta,1}} \leq ||b_1||_{L_{\sigma,1}}\,||f||_{L_{\sigma,1}}\,||\phi||_{G^0_{\sigma,1}}$$

and

$$A(x,D)B(y,D)\phi(x) = AB\left(H_{\alpha,\beta}b_1\right)(x)\left(H_{\alpha,\beta}f_1\right)(x)\phi(x).$$

Proof 6 *By definition (4.1) and (2.1) we have*

$$A(x,D)B(x,D) = \int_0^\infty (xt)^{-(\alpha-\beta)}J_{\alpha-\beta}(xt)\int_0^\infty (x\lambda)^{-(\alpha-\beta)}J_{\alpha-\beta}(x\lambda)b_1(\lambda)b_2(t)\lambda^{4\alpha}d\lambda$$
$$\times\left(H_{\alpha,\beta}B(y,D)\phi(y)\right)(t)t^{4\alpha}dt$$

Now using relation (1.11), we obtain

$$\begin{aligned}
A(x,D)B(x,D) &= \int_0^\infty\int_0^\infty b_1(\lambda)b_2(t)\left(H_{\alpha,\beta}B(y,D)\phi(y)\right)(t)(\lambda t)^{4\alpha}d\lambda dt\\
&\times\frac{1}{\left[2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right]^2}\int_0^\infty (xz)^{-(\alpha-\beta)}J_{\alpha-\beta}(xz)\\
&\times\int_0^\infty\int_0^\infty \left(b_2(t)(f_1\#f_2H_{\alpha,\beta}\phi)\right)(t)b_1(\lambda)D_{\alpha,\beta}(t,\lambda,z)(t\lambda)^{4\alpha}z^{4\alpha}d\lambda dt dz\\
&= \frac{1}{\left[2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right]}\int_0^\infty (xz)^{-(\alpha-\beta)}J_{\alpha-\beta}(xz)\int_0^\infty\int_0^\infty b_2(f_1\#f_2H_{\alpha,\beta}\phi)\\
&\times D_{\alpha,\beta}(t,\lambda,z)\frac{1}{\left[2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right]}b_1(\lambda)\lambda^{4\alpha}t^{4\alpha}z^{4\alpha}d\lambda dt dz\\
&= \int_0^\infty (xz)^{-(\alpha-\beta)}J_{\alpha-\beta}(xz)\int_0^\infty \tau_z\left(b_2(f_1\#f_2H_{\alpha,\beta}\phi)\right)(\lambda)b_1(\lambda)\\
&\times\frac{1}{\left[2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right]}\lambda^{4\alpha}d\lambda z^{4\alpha}dz\\
&= \int_0^\infty (xz)^{-(\alpha-\beta)}J_{\alpha-\beta}(xz)\left[b_1\#b_2(f_1\#f_2H_{\alpha,\beta}\phi)\right](z)z^{4\alpha}dz
\end{aligned}$$

Now an application of the inverse Hankel type transform yields,

$$\int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) A(x,D)B(x,D)\phi(x)x^{4\alpha}dx = \left[b_1 \# b_2(f_1 \# f_2 H_{\alpha,\beta}\phi)\right](z)$$

Therefore

$$\left[H_{\alpha,\beta}A(x,D)B(x,D)\phi(x)\right](z) = \left[b_1 \# A(f_1 \# B H_{\alpha,\beta}\phi)\right](z)$$

Thus

$$\int_0^\infty \left|H_{\alpha,\beta}A(x,D)B(x,D)\phi(x)\right|(z)d\sigma(z) = AB\int_0^\infty \left[|b_1| \# |f_1| \# |H_{\alpha,\beta}\phi|\right](z)d\sigma(z)$$

Hence,

$$\begin{aligned} \left|\left|H_{\alpha,\beta}A(x,D)B(x,D)\phi(x)\right|\right|_{L_{\sigma,1}} &= AB\left|\left||b_1| \# (|f_1| \# |H_{\alpha,\beta}\phi|)\right|\right|_{L_{\sigma,1}} \\ &\le AB\left|\left|b_1\right|\right|_{L_{\sigma,1}} \left|\left|(|f_1| \# |H_{\alpha,\beta}\phi|)\right|\right|_{L_{\sigma,1}} \end{aligned}$$

Now we can use definition (1.1), equations (1.7) and (1.15) to obtain

$$\left|\left|A(x,D)B(x,D)\phi(x)\right|\right|_{G^0_{\alpha,\beta,1}} \le AB\left|\left|b_1\right|\right|_{L_{\sigma,1}} \left|\left|f_1\right|\right|_{L_{\sigma,1}} \left|\left|\phi\right|\right|_{G^0_{\alpha,\beta,1}}$$

Finally from we get

$$\begin{aligned} A(x,D)B(x,D)\phi(x) &= H_{\alpha,\beta}\left[b_1 \# b_2(f_1 \# f_2 H_{\alpha,\beta}\phi)\right](x) \\ &= \left(H_{\alpha,\beta}b_1\right)(x) H_{\alpha,\beta}b_2(f_1 \# f_2 H_{\alpha,\beta}\phi)(x) \\ &= \left(H_{\alpha,\beta}b_1\right)(x) H_{\alpha,\beta}(b_2 f_1)(x) H_{\alpha,\beta}\left(f_2 H_{\alpha,\beta}\phi\right)(x) \\ &= AB\left(H_{\alpha,\beta}b_1\right)(x)\left(H_{\alpha,\beta}f_1\right)(x) H_{\alpha,\beta}\left(H_{\alpha,\beta}\phi\right)(x) \\ &= AB\left(H_{\alpha,\beta}b_1\right)(x)\left(H_{\alpha,\beta}f_1\right)(x)\phi(x). \end{aligned}$$

This completes the proof.

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