Volume 5, No.1, January 2018
Journal of Global Research in Mathematical Archives
UGC Approved Journal
RESEARCH PAPER
Available online at http://www.jgrma.info
ISSN 2320 – 5822

A Bessel type operator and the continuous wavelet transform on the half line

B.B. Waphare and P.D. Pansare
MIT Arts, Commerce and Science College, Alandi (D), Pune, India
balasahebwaphare@gmail.com, pradippansare@gmail.com

Abstract

In this paper we consider a singular differential operator $B_{a,b,n}$ on the half line which generalizes the Bessel operator. We construct and investigate a new continuous wavelet transform on $[0,\infty]$ tied to $B_{a,b,n}$ by using harmonic analysis tools corresponding to $B_{a,b,n}$. Further we this wavelet transform to invert an intervening operator between $B_{a,b,n}$ and the second derivative operator $D_x = \frac{d^2}{dx^2}$.

Keywords: Singular differential operator, generalized wavelets, generalized continuous wavelet transform.

Mathematics subject classification: 42A38, 43A15.

1 Introduction

Recent years wavelet transform has studied by many mathematicians, engineers and researchers. Which has wide applications in engineering, medicals and mathematics. In the classical framework, the notion of wavelet was first introduced by J. Morlet a French petroleum engineer at ELF-Aquitaine in connection with his study of seismic traces. The mathematical foundations were given by A. Grossmann and J. Morlet in [5]. The harmonic analyst Y. Meyer and many other mathematicians become aware of this theory and they recognized many classical results inside it (see [2, 8, 9]). Classical wavelets have wide applications ranging from signal analysis in geophysics and acoustics to quantum theory and pure mathematics (see [3, 4, 7] and the references therein). In this paper we consider the second order singular differential operator on the half line.

$$B_{a,b,n}f(x) = D_x^2 f + \frac{a-b}{x} D_x f - \frac{4n(\frac{a-b-1}{2}+n)}{x^2} f(x), D_x \equiv \frac{d}{dx}$$

where $(a-b) > 0$ and $n = 0, 1, 2, \cdots$ For $n = 0$, we have the following differential operator

$$B_{a,b}f(x) = D_x^2 f + \frac{a-b}{x} D_x f, D_x \equiv \frac{d}{dx},$$

which is referred to as the Bessel type operator of order $\frac{a-b-1}{2}$. A well known harmonic analysis on the half line generated by the Bessel type operator $B_{a,b}$ is expressed by Trimeche in [14].
Following [1], it can be shown that the integral transform

$$f(x) = \frac{2\Gamma(\frac{a-b-1}{2}+2n+1)}{\sqrt{\pi}\Gamma(\frac{4n+a-b}{2})} \int_0^1 f(tx)(1-t^2)^{2n+\frac{a-b-2}{2}} dt$$

is a topological isomorphism between two suitable functional spaces, satisfying the intertwining relation

$$\chi \circ \frac{d^2}{dx^2} = B_{a,b,n} \circ \chi,$$

A completely new commutative harmonic analysis on the half line related to the differential operator $B_{a,b,n}$ was initiated through the intertwining operator χ. Our aim is to extend the classical theory of wavelets to the differential operator $B_{a,b,n}$. More explicitly, we call generalized wavelet each function g in a suitable functional space, satisfying the admissibility condition

$$0 < C_g = \int_0^\infty |\mathcal{F}_{B_{a,b,n}}(g)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty,$$

where $\mathcal{F}_{B_{a,b,n}}$ is the generalized Fourier transform related to $B_{a,b,n}$ given by

$$\mathcal{F}_{B_{a,b,n}}(g)(\lambda) = \int_0^\infty f(x)\phi_\lambda(x)x^{a-b}dx,$$

with $\phi_\lambda(x) = x^{2n} \frac{j_{a-b-1+4n}(\lambda x)}{2}$, where j is the normalized spherical Bessel type function of index ν.

With a single generalized wavelet g we construct by dilation and translation a family of generalized wavelets by putting

$$g_{\alpha,\beta}(x) = \frac{1}{\alpha^{a-b+1+2n}} T^\beta(g_\alpha)(x), \quad \alpha > 0, \beta \geq 0,$$

where $g_\alpha(x) = g(x/\alpha)$ and T^β denote the generalized translation operators tied to the differential operator $B_{a,b,n}$.

The generalized continuous wavelet transform associated with $B_{a,b,n}$ is defined for regular functions f on $[0, \infty)$ by

$$\Phi_g(f)(\alpha, \beta) = \int_0^\infty f(x)\overline{g_{\alpha,\beta}(x)}x^{a-b}dx.$$

2 Preliminaries

In this section we recapitulate some facts about harmonic analysis related to the Bessel type operator $B_{a,b}$. We cite here as briefly as possible, only those properties actually required for the discussion. For more detail we refer to [14].

Here we use the notations:

$$\|\cdot\|_{p,a,b,2n} \equiv \|\cdot\|_{p,\frac{a-b-1}{2}+2n}$$

$$L^p_{a,b,2n} \equiv L^p_{\frac{a-b-1}{2}+2n}$$

$$d\mu_{a,b,2n} \equiv d\mu_{\frac{a-b-1}{2}+2n}, \quad \mu_{a,b,2n} = \mu_{\frac{a-b-1+4n}{2}}$$

$$W_{a,b,2n} \equiv W_{\frac{a-b-1}{2}+2n}$$

$$\tau^x_{a,b,2n} \equiv \tau^x_{\frac{a-b-1}{2}+2n}$$
$$R_{a,b,2n} \equiv R_{\frac{a-b-1}{2}+2n}$$
$$\|\cdot\|_{1,a,b,2n} \equiv \|\cdot\|_{1,\frac{a-b-1}{2}+2n}$$
$$\mathcal{F}_{a,b,2n} \equiv \mathcal{F}_{\frac{a-b-1}{2}+2n}$$
$$C^{a,b,2n}_{\mathcal{M}^{-1}g} \equiv C^{\frac{a-b-1+4n}{2}}_{\mathcal{M}^{-1}g}$$
$$(\mathcal{M}^{-1}g)^{a,b,2n}_{\alpha,\beta} \equiv (\mathcal{M}^{-1}g)^{\frac{a-b-1+4n}{2}}_{\alpha,\beta}$$
$$S^{a,b,2n}_{\mathcal{M}^{-1}g}(\;) \equiv S^{\frac{a-b-1+4n}{2}}_{\mathcal{M}^{-1}g}(\;)$$

Throughout this section assume $(a-b)>0$.
Define $B^p_{a,b}, 1 \leq p < \infty$ as the class of measurable function f on $[0,\infty)$ for which $\|f\|_{p,a,b} < \infty$, where

$$\|f\|_{p,a,b} = \left(\int_0^\infty |f(x)|^p x^{a-b} dx\right)^{1/p}, if \quad p < \infty$$

and

$$\|f\|_{\infty,a,b} = \|f\|_\infty = ess \sup_{x \geq 0} |f(x)|.$$

The Fourier-Bessel type transform of order $\frac{a-b-1}{2}$ is defined for a function $f \in B^1_{a,b}$ by

$$\mathcal{F}_{a,b}(f)(\lambda) = \int_0^\infty f(x) j_{\frac{a-b-1}{2}}(\lambda x) x^{a-b} dx, \lambda \geq 0, \tag{2.1}$$

where $j_{\frac{a-b-1}{2}}$ is the normalized spherical Bessel type function of index $\frac{a-b-1}{2}$ defined by

$$j_{\frac{a-b-1}{2}}(z) = \Gamma(\frac{a-b+1}{2}) \sum_{n=0}^\infty \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(\frac{2n+a-b+1}{2})}, z \in \mathcal{C} \tag{2.2}$$

Proposition 2.1. *(i) The Fourier-Bessel type transform $\mathcal{F}_{a,b}$ maps continuously and invectively $L^1_{a,b}$ into the space $C_0([0,\infty))$ (of continuous function on $[0,\infty)$ vanishing at infinity).*

(ii) If both f and $\mathcal{F}_{a,b}(f)$ are in $L^1_{a,b}$ then

$$f(x) = \int_0^\infty \mathcal{F}_{a,b}(f)(\lambda) j_{\frac{a-b-1}{2}}(\lambda x) d\mu_{a,b}(\lambda),$$

for almost all $x \geq 0$, where

$$d\mu_{a,b}(\lambda) = \frac{1}{4^{\frac{a-b-1}{2}} \left(\Gamma(\frac{a-b+1}{2})\right)^2} \lambda^{a-b} d\lambda \tag{2.3}$$

(iii) For every $f \in L^1_{a,b} \bigcap L^2_{a,b}$, *we have*

$$\int_0^\infty |f(x)|^2 x^{a-b} dx = \int_0^\infty |\mathcal{F}_{a,b}(f)(\lambda)|^2 d\mu_{a,b}(\lambda).$$

(iv) The Fourier-Bessel type transform $\mathcal{F}_{a,b}$ *extends uniquely to an isometric isomorphism from* $L^2_{a,b}$ *onto* $L^2([0,\infty),\mu_{a,b})$. *The inverse transform is given by*

$$\mathcal{F}^{-1}_{a,b}(g)(x) = \int_0^\infty g(\lambda) j_{\frac{a-b-1}{2}}(\lambda x) d\mu_{a,b}(\lambda),$$

where the integral converges in $L^2_{a,b}$.

The Bessel type translation operators $\tau^x_{a,b}, x \geq 0$ are defined by

$$\tau^x_{a,b}(f)(y) = \alpha_{a,b} \int_0^\pi f(\sqrt{x^2+y^2+2xy\cos\theta})(\sin\theta)d\theta, \tag{2.4}$$

where

$$\alpha_{a,b} = \frac{2\Gamma(\frac{a-b+1}{2})}{\sqrt{\pi}\Gamma(\frac{a-b}{2})} \tag{2.5}$$

For $x, y > 0$, a change of variables yields

$$\tau^x_{a,b}(f)(y) = \int_{|x-y|}^{x+y} f(z) W_{a,b}(x,y,z) z^{a-b} dz, \tag{2.6}$$

with

$$W_{a,b}(x,y,z) = \frac{2^{\frac{3-a+b}{2}}[\Gamma(\frac{a-b+1}{2})]^2}{\sqrt{\pi}\Gamma(\frac{a-b}{2})} \frac{[(x+y)^2 - z^2]^{\frac{a-b-2}{2}}[z^2-(x-y)^2]^{\frac{a-b-2}{2}}}{(xyz)^{a-b-1}} \tag{2.7}$$

The Bessel type convolution product of two functions f, g on $[0,\infty)$ is defined by the relation.

$$f \star_{a,b} g(x) = \int_0^\infty \tau^x_{a,b} f(y) g(y) y^{a-b} dy, x \geq 0. \tag{2.8}$$

Proposition 2.2. *(i) Let* $p \in [1,\infty)$ *and* $f \in L^p_{a,b}$. *Then for all* $x \geq 0, \tau^x_{a,b} f \in L^p_{a,b}$ *and*

$$\|\tau^x_{a,b} f\|_{p,a,b} \leq \|f\|_{p,a,b}.$$

(ii) Let $p, q \in [1,\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in L^p_{a,b}$ *and* $g \in L^q_{a,b}$, *then for every* $x \geq 0$ *we have*

$$\int_0^\infty \tau^x_{a,b} f(y) g(y) y^{a-b} dy = \int_0^\infty f(y) \tau^x_{a,b} g(y) y^{a-b} dy.$$

(iii) Let $p, q, r \in [1,\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. *If* $f \in L^p_{a,b}$ *and* $g \in L^q_{a,b}$, *then* $f \star_{a,b} g \in L^r_{a,b}$ *and*

$$\|f \star_{a,b} g\|_{r,a,b} \leq \|f\|_{p,a,b} \|g\|_{q,a,b}.$$

(iv) *For* $f \in L^1_{a,b}$ *and* $g \in L^p_{a,b}$, $p = 1$ *or* 2, *we have*

$$\mathcal{F}_{a,b}(f \star_{a,b} g) = \mathcal{F}_{a,b}(f)\mathcal{F}_{a,b}(g).$$

Definition 2.1. *We say that a function* $g \in L^2_{a,b}$ *is a Bessel type wavelet of order* $\frac{a-b-1}{2}$ *if it satisfies the admissibility condition*

$$0 < C^{a,b}_g = \int_0^\infty |\mathcal{F}_{\alpha,\beta}(g)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty. \tag{2.9}$$

Definition 2.2. *Let* $g \in L^2_{a,b}$ *be a Bessel type wavelet of order* $\frac{a-b-a}{2}$. *The Bessel type continuous wavelet transform is defined for suitable functions* f *on* $[0, \infty)$ *by*

$$S^{a,b}_g(f)(\alpha, \beta) = \int_0^\infty f(x) g^{a,b}_{\alpha,\beta}(x) x^{a-b} dx, \tag{2.10}$$

where $\alpha > 0, \beta \geq 0$,

$$g^{a,b}_{\alpha,\beta}(x) = \frac{1}{a^{a-b+1}} \tau^b_{a,b}(g_\alpha)(x), \tag{2.11}$$

and

$$g_\alpha(x) = g(x/a). \tag{2.12}$$

The Bessel continuous wavelet transform has been investigated in depth in [14] from which we call the following basic properties.

Theorem 2.1. *Let* $g \in L^2_{a,b}$ *be a Bessel type wavelet type wavelet of order* $\frac{a-b-1}{2}$. *Then*

(i) *For all* $f \in L^2_{a,b}$, *we have the Planchevel formula*

$$\int_0^\infty |f(x)|^2 x^{a-b} dx = \frac{1}{C^{a,b}_g} \int_0^\infty \int_0^\infty |S^{a,b}_g(f)(\alpha, \beta)|^2 \beta^{a-b} d\beta \frac{d\alpha}{\alpha}$$

(ii) *Assume that* $\|\mathcal{F}_{a,b}(g)\|_\infty < \infty$. *For* $f \in L^2_{a,b}$ *and* $0 < \epsilon < \delta < \infty$, *the function*

$$f^{\epsilon,\delta}(x) = \frac{1}{C_g} \int_\epsilon^\delta \int_0^\infty S^{a,b}_g(f)(\alpha, \beta) g^{a,b}_{\alpha,\beta}(x) \beta^{a-b} d\beta \frac{d\alpha}{\alpha},$$

belongs to $L^2_{a,b}$ *and satisfies*

$$\lim_{\epsilon \to 0} \lim_{\delta \to 0} \|f^{\epsilon,\delta} - f\|_{2,a,b} = 0.$$

(iii) *For* $f \in L^1_{a,b}$ *such that* $\mathcal{F}_{a,b}(f) \in L^1_{a,b}$, *we have*

$$f(x) = \frac{1}{C^{a,b}_g} \int_0^\infty \left(\int_0^\infty S^{a,b}_g(f)(\alpha, \beta) g^{a,b}_{\alpha,\beta}(x) \beta^{a-b} d\beta \right) \frac{d\alpha}{\alpha},$$

for almost all $x \geq 0$.

3 Harmonic analysis associated with $B_{a,b,n}$

From now onward, assume $(a-b)>0$ and $n=0,1,2,\cdots$. Let $\mathcal{M}$ be the map defined by

$$\mathcal{M}f(x)=x^{2n}f(x).$$

Let $L^p_{a,b,n},\ 1\le p<\infty$ be the class of measurable functions f on $[0,\infty)$ for which $\|f\|_{p,a,b,n}=\|\mathcal{M}^{-1}f\|_{p,a,b,2n}<\infty$.

Remark 3.1. *We notice that $\mathcal{M}$ is an isometric from $L^p_{a,b,2n}$ onto $L^p_{a,b,2n}$.*

3.1 Generalized Fourier transform

Let $\lambda\in\mathbb{C}$ and $x\in\mathbb{R}$, put

$$\phi_\lambda(x)=x^{2n}j_{\frac{a-b-1+4n}{2}}(\lambda x), \tag{3.1}$$

where $j_{\frac{a-b-1+4n}{2}}$ is the normalized Bessel type function of index $\frac{a-b-1+4n}{2}$ given by (2.2). From [1] recall the following properties.

Proposition 3.1. *(i) ϕ_λ possesses the Laplace type integral representation*

$$\phi_\lambda(x)=\alpha_{a,b,2n}x^{2n}\int_0^1\cos(\lambda tx)(1-t^2)^{\frac{a-b-2+4n}{2}}dt, \tag{3.2}$$

where $\alpha_{a,b,2n}$ is given by (2.5). (Note that here $\alpha_{a,b,2n}$ means $\alpha_{\frac{a-b-1}{2}+2n}$)

(ii) ϕ_λ satisfies the differential equation

$$B_{a,b,n}\phi_\lambda=-\lambda^2\phi_\lambda,$$

(iii) For all $\lambda\in\mathbb{C}$ and $x\in\mathbb{R}$,

$$|\phi_\lambda(x)|\le x^{2n}e^{|Im\lambda||x|}.$$

Definition 3.1. *The generalized Fourier transform is defined for a function $f\in L^1_{a,b,n}$ by*

$$\mathcal{F}_{B_{a,b,n}}(f)(\lambda)=\int_0^\infty f(x)\phi_\lambda(x)x^{a-b}dx,\lambda\ge 0 \tag{3.3}$$

Remark 3.2. *(i) By (3.1) and (3.3) observe that*

$$\mathcal{F}_{B_{a,b,n}}=\mathcal{F}_{a,b,2n}\circ\mathcal{M}^{-1}\quad(\mathcal{F}_{a,b,2n}=\mathcal{F}_{\frac{a-b-1}{2}+2n}) \tag{3.4}$$

where $\mathcal{F}_{a,b,2n}$ is the Fourier-Bessel type transform of order $\frac{a-b-1+4n}{2}$ given by (2.1).

(ii) If $f\in L^1_{a,b,n}$, then $\mathcal{F}_{B_{a,b,n}}(f)\in C_0([0,\infty))$ and

$$\|\mathcal{F}_{B_{a,b,n}}(f)\|_\infty\le\|f\|_{1,a,b,n}.$$

Theorem 3.1. *Let $f \in L^1_{a,b,n}$ such that $\mathcal{F}_{B_{a,b,n}}(f) \in L^1_{a,b,2n}$ (here $L^1_{a,b,2n} = L^1_{\frac{a-b-1}{2}+2n}$). Then for almost all $x \geq 0$,*

$$f(x) = \int_0^\infty \mathcal{F}_{B_{a,b,n}}(f)(\lambda)\phi_\lambda(x)d\mu_{a,b,2n}(\lambda),$$

where $\mu_{a,b,2n}$ is given by (2.3).

Proof. From (3.1), (3.4) and Proposition 2.1(ii), we have

$$\begin{aligned}\int_0^\infty \mathcal{F}_{B_{a,b,n}}(f)(\lambda)\phi_\lambda(x)d\mu_{a,b,2n}(\lambda) &= x^{2n}\int_0^\infty \mathcal{F}_{a,b,2n}(\mathcal{M}^{-1}f)(\lambda)j_{\frac{a-b-1+4n}{2}}(\lambda x)d\mu_{a,b,2n}(\lambda) \\ &= x^{2n}\mathcal{M}^{-1}f(x) \\ &= f(x), \quad \textit{for almost all} \;\; x \geq 0.\end{aligned}$$

Thus proof is completed. □

Theorem 3.2. *(i) For every $f \in L^1_{a,b,n}\bigcap L^2_{a,b,n}$, we have the Plancherel formula*

$$\int_0^\infty |f(x)|^2 x^{a-b}dx = \int_0^\infty |\mathcal{F}_{B_{a,b,n}}(f)(\lambda)|^2 d\mu_{a,b,2n}(\lambda)$$

(Here $d\mu_{a,b,2n} \equiv d\mu_{\frac{a-b-1}{2}+2n}$)

(ii) The generalized Fourier transform $\mathcal{F}_{B_{a,b,n}}$ extends uniquely to an isometric isomorphism from $L^2_{a,b,n}$ onto $L^2([0,\infty),\mu_{a,b,2n})$. The inverse transform is given by

$$\mathcal{F}^{-1}_{B_{a,b,n}}(g)(x) = \int_0^\infty g(\lambda)\phi_\lambda(x)d\mu_{a,b,2n}(\lambda),$$

where the integral converges in $L^2_{a,b,n}$.

Proof. Let $f \in L^1_{a,b,n}\bigcap L^2_{a,b,n}$. By (3.4) and proposition 2.1(iii), we have

$$\begin{aligned}\int_0^\infty |\mathcal{F}_{B_{a,b,n}}(f)(\lambda)|^2 d\mu_{a,b,2n}(\lambda) &= \int_0^\infty |\mathcal{F}_{a,b,2n}(\mathcal{M}^{-1}f)(\lambda)|^2 d\mu_{a,b,2n}(\lambda) \\ &= \int_0^\infty |\mathcal{M}^{-1}f(x)|^2 x^{a-b+4n}dx \\ &= \int_0^\infty |f(x)|^2 x^{a-b}dx\end{aligned}$$

which gives (i).
The proof of (ii) is standard can be proved easily.
Thus proof is completed. □

3.2 Generalized convolution product

Definition 3.2. *We define the generalized translation operators* T^x, $x \geq 0$ *by the relation*

$$T^x f(y) = (xy)^{2n} \tau^x_{a,b,2n}(\mathcal{M}^{-1} f)(y), \ y \geq 0, \tag{3.5}$$

(Here $\tau^x_{a,b,2n} \equiv \tau^x_{\frac{a-b-1}{2}+2n}$*)*
where $\tau^x_{a,b,2n}$ *are the Bessel type translation operators of order* $\frac{a-b-1+4n}{2}$ *given by (2.4).*

Remark 3.3. *Assume that* $x, y > 0$*. Then according to (2.6) and (3.5) we have*

$$T^x(f)(y) = \int_{|x-y|}^{x+y} f(z) W_{a,b,n}(x,y,z) z^{a-b} dz,$$

with $W_{a,b,n} = (xyz)^{2n} W_{a,b,2n}(x,y,z)$*, where* $W_{a,b,2n}(x,y,z)$ *is given by (2.7). (Here* $W_{a,b,2n} = W_{\frac{a-b-1}{2}+2n}$*).*

Definition 3.3. *The generalized convolution product of two functions* f *and* g *on* $[0,\infty)$ *is defined by*

$$f\#g(x) = \int_0^\infty T^x f(y) g(y) y^{a-b} dy, \ x \geq 0. \tag{3.6}$$

Remark 3.4. *Note that by (3.5) we have*

$$f\#g = \mathcal{M}[(\mathcal{M}^{-1} f) \star_{a,b,2n} (\mathcal{M}^{-1} g)], \tag{3.7}$$

where $\star_{a,b,2n}$ *is the Bessel convolution given by (2.8).*

Proposition 3.2. *(i) Let* $f \in L^p_{a,b,n}$, $1 \leq p < \infty$*. Then for all* $x \geq 0$*, the function* $T^x f \in L^p_{a,b,n}$*, and*

$$\|T^x f\|_{p,a,b,n} \leq x^{2n} \|f\|_{p,a,b,n}.$$

(ii) For $f \in L^p_{a,b,n}$, $p = 1$ *or* 2*, we have*

$$\mathcal{F}_{B_{a,b,n}}(T^x f)(\lambda) = \phi_\lambda(x) \mathcal{F}_{B_{a,b,n}}(f)(\lambda).$$

(iii) Let $p, q \in [1,\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. If* $f \in L^p_{a,b,n}$ *and* $g \in L^p_{a,b,n}$*, then*

$$\int_0^\infty T^x f(y) g(y) y^{a-b} dy = \int_0^\infty f(y) T^x g(y) y^{a-b} dy.$$

(iv) Let $p, q, r \in [1,\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$*. If* $f \in L^p_{a,b,n}$ *and* $g \in L^q_{a,b,n}$ *then* $f\#g \in L^r_{a,b,n}$ *and*

$$\|f\#g\|_{r,a,b,n} \leq \|f\|_{p,a,b,n} \|g\|_{q,a,b,n}.$$

(v) For $f \in L^1_{a,b,n}$ and $g \in L^p_{a,b,n}$, $p = 1$ or 2, we have

$$\mathcal{F}_{B_{a,b,n}}(f\#g) = \mathcal{F}_{B_{a,b,n}}(f)\mathcal{F}_{B_{a,b,n}}(g).$$

Proof. (i) Using Proposition 2.2(i) and (3.7), we have

$$\begin{aligned}\|T^x f\|_{p,a,b,n} &= x^{2n}\|\mathcal{M}\circ\tau^x_{a,b,2n}\circ\mathcal{M}^{-1}(f)\|_{p,a,b,n}\\ &= x^{2n}\|\tau^x_{a,b,2n}\circ\mathcal{M}^{-1}(f)\|_{p,a,b,2n}\\ &\leq x^{2n}\|\mathcal{M}^{-1}f\|_{p,a,b,2n}\\ &= x^{2n}\|f\|_{p,a,b,n}.\end{aligned}$$

(ii) Using Proposition 2.2(ii) and (3.1), (3.4) and (3.5), we have

$$\begin{aligned}\mathcal{F}_{B_{a,b,n}}(T^x f)(\lambda) &= \mathcal{F}_{a,b,2n}\circ\mathcal{M}^{-1}\circ\tau^x(f)(\lambda)\\ &= x^{2n}\mathcal{F}_{a,b,2n}\circ\tau^x_{a,b,2n}\circ\mathcal{M}^{-1}(f)(\lambda)\\ &= x^{2n}j_{\frac{a-b-a+4n}{2}}(\lambda x)\mathcal{F}_{a,b,2n}\circ\mathcal{M}^{-1}(f)(\lambda)\\ &= \phi_\lambda(x)\mathcal{F}_{B_{a,b,n}}(f)(\lambda).\end{aligned}$$

(iii) Using Proposition 2.2(iii) and (3.5), we have

$$\begin{aligned}\int_0^\infty T^x f(y)g(y)y^{a-b}dy &= x^{2n}\int_0^\infty \tau^x_{a,b,2n}(\mathcal{M}^{-1}f)(y)\mathcal{M}^{-1}(g)(y)y^{a-b+4n}dy\\ &= x^{2n}\int_0^\infty \mathcal{M}^{-1}f(y)\tau^x_{a,b,2n}(\mathcal{M}^{-1}g)(y)y^{a-b+4n}dy\\ &= \int_0^\infty f(y)T^x g(y)y^{a-b}dy.\end{aligned}$$

(iv) By using Proposition 2.2(iv) and (3.7), we have

$$\begin{aligned}\|f\#g\|_{r,a,b,n} &= \|(\mathcal{M}^{-1}f)\star_{a,b,2n}(\mathcal{M}^{-1}g)\|_{r,a,b,2n}\\ &\leq \|\mathcal{M}^{-1}f\|_{p,a,b,2n}\|\mathcal{M}^{-1}g\|_{q,a,b,2n}\\ &= \|f\|_{p,a,b,n}\|g\|_{q,a,b,n}.\end{aligned}$$

(v) By Proposition 2.2(v) and using (3.4) and (3.7), we have

$$\begin{aligned}\mathcal{F}_{B_{a,b,n}}(f\#g) &= \mathcal{F}_{a,b,2n}[(\mathcal{M}^{-1}f)\star_{a,b,2n}(\mathcal{M}^{-1}g)]\\ &= \mathcal{F}_{a,b,2n}(\mathcal{M}^{-1}f)\mathcal{F}_{a,b,2n}(\mathcal{M}^{-1}g)\\ &= \mathcal{F}_{B_{a,b,n}}(f)\mathcal{F}_{B_{a,b,n}}(g).\end{aligned}$$

Thus proof is completed. □

3.3 Transmutation operators

We denote by $\epsilon(\mathbb{R})$ the space of C^∞ even functions on $\mathbb{R}$, provided with the topology of compact convergence for all derivatives. For $\alpha > 0$, $D_\alpha(\mathbb{R})$ denotes the space C^∞ even functions on $\mathbb{R}$ which are supported in $[-a, a]$, equipped with the topology induced by $\epsilon(\mathbb{R})$. Put $D(\mathbb{R}) = \bigcup_{\alpha>0} D_\alpha(\mathbb{R})$ endowed with the inductive limit topology. Let $\epsilon_n(\mathbb{R})$ (resp. $D_n(\mathbb{R})$) stand for the subspace of $\epsilon(\mathbb{R})$ (resp. $D(\mathbb{R})$) consisting of functions f such that $f(0) = \cdots = f^{(2n-1)}(0) = 0$.

Definition 3.4. *For a locally bounded function f on $[0, \infty)$, we define the integral transform χ by*

$$\chi f(x) = \alpha_{a,b,2n} x^{2n} \int_0^1 f(tx)(1-t^2)^{\frac{a-b-2+4n}{2}} dt \tag{3.8}$$

where $\alpha_{a,b,2n}$ is given by (2.5). (Here $\alpha_{a,b,2n} = \alpha_{\frac{a-b-1+4n}{2}}$)

Remark 3.5. *(i) For $n = 0$, χ reduces to the Riemann-Liouville integral transform of order $\frac{a-b-1}{2}$ given by*

$$R_{a,b}(f)(x) = \alpha_{a,b} \int_0^1 f(tx)(1-t^2)^{\frac{a-b-2}{2}} dt, \; x \geq 0.$$

(ii) It is easily check that

$$\chi = \mathcal{M} \circ R_{a,b,2n} \tag{3.9}$$

(iii) From (3.2) and (3.8), we have

$$\phi_\lambda(x) = \chi(\cos(\lambda))(x). \tag{3.10}$$

Definition 3.5. *We define the integral transform ${}^t\chi$ for a smooth function f on $[0, \infty)$ by*

$${}^t\chi f(y) = \alpha_{a,b,2n} \int_y^\infty f(x)(x^2-y^2)^{\frac{a-b-3+4n}{2}} \frac{dx}{x^{2n-1}}.$$

Remark 3.6. *(i) For $n = 0$, ${}^t\chi$ is just the Weyl integral transform of order $(\frac{a-b-1}{2})$ given by*

$$W_{a,b}(f)(y) = \alpha_{a,b} \int_y^\infty f(x)(x^2-y^2)^{\frac{a-b-2}{2}} x dx, \; y \geq 0,$$

(ii) It is easily seen that

$${}^t\chi = W_{a,b,2n} \circ \mathcal{M}^{-1}. \tag{3.11}$$

Proposition 3.3. *(i) If $f \in L^\infty([0,\infty), dx)$ then $\chi f \in L^\infty_{a,b,n}$ and $\|\chi f\|_{\infty,a,b,n} \leq \|f\|_\infty$.*

(ii) If $f \in L^1_{a,b,n}$ then ${}^t\chi f \in L^1([0,\infty), dx)$ and $\|{}^t\chi f\|_1 \leq \|f\|_{1,a,b,n}$.

(iii) For every $f \in L^\infty([0,\infty), dx)$ and $g \in L^1_{a,b,n}$, we have the duality relation

$$\int_0^\infty \chi f(x) g(x) x^{a-b} dx = \int_0^\infty f(y)\ {}^t\chi g(y) dy$$

(iv) For all $f \in L^1_{a,b,n}$, we have

$$\mathcal{F}_\Delta(f) = \mathcal{F}_c \circ\ {}^t\chi(f), \tag{3.12}$$

where $\mathcal{F}_c$ is the cosine transform given by

$$\mathcal{F}_c(f)(\lambda) = \int_0^\infty f(x) \cos(\lambda x) dx,\ \ \lambda \geq 0$$

(v) Let $f, g \in L^1_{a,b,n}$. Then

$${}^t\chi(f \# g) =^t \chi f \star\ {}^t\chi g,$$

where $\star$ is the symmetric convolution product on $[0,\infty)$ defined by

$$h_1 \star h_2(x) = \int_0^\infty \sigma_x(h_1)(y) h_2(y) dy,$$

with

$$\sigma_x(h_1)(y) = \frac{1}{2}[h_1(x+y) + h_1(|x-y|)].$$

(vi) Let $f \in L^1_{a,b,n}$ and $g \in L^\infty([0,\infty), dx)$. Then

$$\chi({}^t\chi f \star g) = f \#(\chi g).$$

Proof. (i) By (3.9) and [[14], Equation (2.I.48)], we have

$$\|\chi f\|_{\infty,a,b,n} = \|R_{a,b,2n} f\|_\infty \leq \|f\|_\infty.$$

(ii) By (3.11) and [[14], Equation (2.II.3)], we have

$$\|{}^t\chi f\|_1 \leq \|\mathcal{M}^{-1} f\|_{1,a,b,2n} = \|f\|_{1,a,b,n}.$$

(iii) By (3.9), (3.11) and [[14], Equation (2.II.2)], we have

$$\begin{aligned}
\int_0^\infty \chi f(x) g(x) x^{a-b} dx &= \int_0^\infty R_{a,b,2n}(f)(x) \mathcal{M}^{-1} g(x) x^{a-b+4n} dx \\
&= \int_0^\infty f(y) W_{a,b,2n}(\mathcal{M}^{-1} g)(y) dy \\
&= \int_0^\infty f(y)\ {}^t\chi g(y) dy.
\end{aligned}$$

(iv) By (3.4), (3.11) and [[14], Equation (5.II.14)], we have

$$\begin{aligned}\mathcal{F}_c \circ {}^t\chi(f) &= \mathcal{F}_c \circ W_{a,b,2n} \circ \mathcal{M}^{-1}(f)\\ &= \mathcal{F}_{a,b,2n} \circ \mathcal{M}^{-1}(f)\\ &= \mathcal{F}_\Delta(f).\end{aligned}$$

(v) By (3.7), (3.11) and [[14], Equation (5.II.15)], we have

$$\begin{aligned}{}^t\chi(f\#g) &= W_{a,b,2n}[(\mathcal{M}^{-1}f) \star_{a,b,2n} (\mathcal{M}^{-1}g)]\\ &= (W_{a,b,2n}\mathcal{M}^{-1}f) \star (W_{a,b,2n}\mathcal{M}^{-1}g)\\ &= {}^t\chi f \star {}^t\chi g.\end{aligned}$$

(vi) By (3.7), (3.9), (3.11) and [[14], Equation (7.IV.9)], we have

$$\begin{aligned}f\#(\chi g) &= \mathcal{M}[(\mathcal{M}^{-1}f) \star_{a,b,2n} (\mathcal{M}^{-1}\chi g)]\\ &= \mathcal{M}[(\mathcal{M}^{-1}f) \star_{a,b,2n} (R_{a,b,2n}g)]\\ &= \mathcal{M}R_{a,b,2n}[(W_{a,b,2n}\mathcal{M}^{-1}f) \star g]\\ &= \chi({}^t\chi f \star g).\end{aligned}$$

Thus proof is completed. □

χ and ${}^t\chi$ are intervening operators between Δ and the second derivative operator $\frac{d^2}{dx^2}$ by virtue of the following theorem in [1].

Theorem 3.3. *(i) The integral transform χ is an isomorphism from $\epsilon(\mathbb{R})$ onto $\epsilon_n(\mathbb{R})$ satisfying the intervening relation*

$$\chi \circ \frac{d^2}{dx^2}(f) = \Delta \circ \chi(f), \;\; f \in \epsilon(\mathbb{R}).$$

(ii) The integral transform ${}^t\chi$ is an isomorphism from $D_n(\mathbb{R})$ onto $D(\mathbb{R})$ satisfying the intervening relation

$$\frac{d^2}{dx^2} \circ {}^t\chi(f) ={}^t \chi \circ \Delta(f), \;\; f \in D_n(\mathbb{R}).$$

4 Generalized Wavelets

In this section we obtain Plancherel formula, Calderm's formula and inversion formula for generalized wavelet.

Definition 4.1. *A generalized wavelet is a function $g \in L^2_{a,b,n}$ satisfying the admissibility condition*

$$0 < C_g = \int_0^\infty |\mathcal{F}_\Delta(g)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty \tag{4.1}$$

Remark 4.1. *(i) Let $0 \neq g \in L^2_{a,b,n}$ satisfying the condition that there exist $\eta > 0$ such that $\mathcal{F}_\Delta(g)(\lambda) - \mathcal{F}_\Delta(g)(0) = O(\lambda^n)$, as $\lambda \to 0$. Then (4.1) is equivalent to $\mathcal{F}_\Delta(g)(0) = 0$.*

(ii) By (2.9), (3.4) and (4.1), $g \in L^2_{a,b,n}$ is a generalized wavelet if and only if $\mathcal{M}^{-1}g$ is a Bessel wavelet of order $\frac{a-b-1+4n}{2}$, and we have

$$C_g = C_{\mathcal{M}^{-1}g} \tag{4.2}$$

Note: For $g \in L^2_{a,b,n}$ and $(a,b) \in (0,\infty) \times [0,\infty)$, put

$$g_{\alpha,\beta}(x) = \frac{1}{\alpha^{a-b+2n+1}} T^\beta(g_\alpha)(x) \tag{4.3}$$

where g_α is given by (2.12) and T^β are the generalized translation operators defined by (3.5).

Proposition 4.1. *For all $\alpha > 0$ and $\beta > 0$, we have*

$$g_{\alpha,\beta}(x) = (\beta x)^{2n} (\mathcal{M}^{-1})^{a,b,2n}_{\alpha,\beta}(x) \tag{4.4}$$

Proof. By making use of (2.11), (3.5) and (4.3), we have

$$\begin{aligned} g_{\alpha,\beta}(x) &= \frac{1}{\alpha^{a-b+2n+1}} T^\beta(g_\alpha)(x) \\ &= \frac{(\beta x)^{2n}}{\alpha^{a-b+2n+1}} \tau^\beta_{a,b,2n}(\mathcal{M}^{-1}g_\alpha)(x) \\ &= \frac{(\beta x)^{2n}}{\alpha^{a-b+4n+1}} \tau^\beta_{a,b,2n}(\mathcal{M}^{-1}g)_\alpha(x) \\ &= (\beta x)^{2n} (\mathcal{M}^{-1}g)^{a,b,2n}_{\alpha,\beta}(x). \end{aligned}$$

Thus proof is completed. □

Definition 4.2. *Let $g \in L^2_{a,b,n}$ be a generalized wavelet. We define for regular functions f on $[0,\infty)$, the generalized continuous wavelet transform by*

$$\Phi_g(f)(\alpha,\beta) = \int_0^\infty f(x)\overline{g_{\alpha,\beta}(x)} x^{a-b} dx, \tag{4.5}$$

which can also be written in the form

$$\Phi_g(f)(\alpha,\beta) = \frac{1}{\alpha^{a-b+2n+1}} f \# \overline{g_\alpha}(\beta), \tag{4.6}$$

where # is the generalized convolution product given by (3.6).

Proposition 4.2. *We have*

$$\Phi_g(f)(\alpha,\beta) = \beta^{2n} S^{a,b,2n}_{\mathcal{M}^{-1}g}(\mathcal{M}^{-1}f)(\alpha,\beta) \tag{4.7}$$

Proof. From (2.10), (4.4) and (4.5), we deduce that

$$\begin{aligned}\Phi_g(f)(\alpha,\beta) &= \int_0^\infty f(x)\overline{g_{\alpha,\beta}(x)}x^{a-b}dx \\ &= \beta^{2n}\int_0^\infty (\mathcal{M}^{-1}f)(x)\overline{(\mathcal{M}^{-1}g)^{a,b,2n}_{\alpha,\beta}(x)}x^{a-b+4n}dx \\ &= \beta^{2n}S^{a,b,2n}_{\mathcal{M}^{-1}g}(\mathcal{M}^{-1}f)(\alpha,\beta).\end{aligned}$$

Thus proof is completed. □

Theorem 4.1. *(Plancherel formula) Let $g \in L^2_{a,b,n}$ be a generalized wavelet. For every $f \in L^2_{a,b,n}$, we have the Plancherel formula*

$$\int_0^\infty |f(x)|^2 x^{a-b}dx = \frac{1}{C_g}\int_0^\infty\int_0^\infty |\Phi_g(f)(\alpha,\beta)|^2\beta^{a-b}d\beta\frac{d\alpha}{\alpha}$$

Proof. By using Theorem 2.1(i) and (4.2) and (4.7), we have

$$\begin{aligned}\int_0^\infty\int_0^\infty |\Phi_g(f)(\alpha,\beta)|^2\beta^{a-b}d\beta\frac{d\alpha}{\alpha} &= \int_0^\infty\int_0^\infty |S^{a,b,2n}_{\mathcal{M}^{-1}g}(\mathcal{M}^{-1}f)(\alpha,\beta)|^2\beta^{a-b+4n}d\beta\frac{d\alpha}{\alpha} \\ &= C^{a,b,2n}_{\mathcal{M}^{-1}g}\int_0^\infty |\mathcal{M}^{-1}f(x)|^2 x^{a-b+4n}dx \\ &= C_g\int_0^\infty |f(x)|^2x^{a-b}dx.\end{aligned}$$

This completes the proof. □

Theorem 4.2. *(Calderm's formula) Let $g \in L^2_{a,b,n}$ be a generalized wavelet such that $\|\mathcal{F}_\Delta(g)\|_\infty < \infty$. Then $f \in L^2_{a,b,n}$ and $0 < \epsilon < \delta < \infty$, the function*

$$f^{\epsilon,\delta}(x) = \frac{1}{C_g}\int_\epsilon^\delta\int_0^\infty \Phi_g(f)(\alpha,\beta)g_{\alpha,\beta}(x)\beta^{a-b}d\beta\frac{d\alpha}{\alpha} \in L^2_{a,b,n}$$

and satisfies

$$\lim_{\substack{\epsilon\to 0\\ \delta\to 0}} \|f^{\epsilon,\delta} - f\|_{2,a,b,n} = 0.$$

Proof. By using (4.2), (4.4) and (4.7), we have

$$f^{\epsilon,\delta}(x) = \frac{x^{2n}}{C^{a,b,2n}_{\mathcal{M}^{-1}g}}\int_\epsilon^\delta\int_0^\infty S^{a,b,2n}_{\mathcal{M}^{-1}g}(\mathcal{M}^{-1}f)(\alpha,\beta)(\mathcal{M}^{-1}g)^{a,b,2n}_{\alpha,\beta}\beta^{a-b+4n}d\beta\frac{d\alpha}{\alpha}.$$

The result follows from Theorem 2.1(ii). Thus proof is completed. □

Theorem 4.3. *(Inversion formula) Let $g \in L^2_{a,b,n}$ be a generalized wavelet. If $f \in L^1_{a,b,n}$ and $\mathcal{F}_\Delta(f) \in L^1_{a,b,2n}$ then we have*

$$f(x) = \frac{1}{C_g} \int_0^\infty \left(\int_0^\infty \Phi_g(f)(\alpha, \beta) g_{\alpha,\beta}(x) \beta^{a-b} d\beta \right) \frac{d\alpha}{\alpha}$$

for almost all $x \geq 0$.

Proof. By using (4.2), (4.4) and (4.7), we have

$$\begin{aligned} & \frac{1}{C_g} \int_0^\infty \left(\int_0^\infty \Phi_g(f)(\alpha, \beta) g_{\alpha,\beta}(x) \beta^{a-b} d\beta \right) \frac{d\alpha}{\alpha} \\ = \quad & \frac{x^{2n}}{C^{a,b,2n}_{\mathcal{M}^{-1}g}} \int_\epsilon^\delta \left(\int_0^\infty S^{a,b,2n}_{\mathcal{M}^{-1}g}(\mathcal{M}^{-1}f)(\alpha, \beta) (\mathcal{M}^{-1}g)^{a,b,2n}_{\alpha,\beta} \beta^{a-b+4n} d\beta \right) \frac{d\alpha}{\alpha}. \end{aligned}$$

Now we can use Theorem 2.1(iii) to complete the proof.
Thus proof is completed. □

5 Inversion of the intertwining operator ${}^t\chi$ through the generalized wavelet transform

In this section we obtain inversion formulas for ${}^t\chi$ involving generalized wavelets. But before this we need to prove some preliminary lemmas.

Lemma 5.1. *Let $0 \neq g \in L^1 \bigcap L^2([0, \infty), dx)$ such that $\mathcal{F}_c(g) \in L^1([0, \infty), dx)$ and satisfying there exists $\eta > \frac{a-b-1+4n}{2}$ such that*

$$\mathcal{F}_c(g)(\lambda) = o(\lambda^n) \tag{5.1}$$

as $\lambda \to 0$. Then $\chi g \in L^2_{a,b,n}$ and

$$\mathcal{F}_\Delta(\chi g)(\lambda) = \frac{2^{a-b+4n} (\Gamma(\frac{a-b+4n+1}{2}))^2}{\pi \lambda^{a-b+4n}} \mathcal{F}_c(g)(\lambda).$$

Proof. We have

$$g(x) = \frac{2}{\pi} \int_0^\infty \mathcal{F}_c(g)(\lambda) \cos(\lambda x) d\lambda$$

Now, by (3.10), we have

$$\chi g(x) = \int_0^\infty h(\lambda) \phi_\lambda(x) d\mu_{a,b,2n}(\lambda), \tag{5.2}$$

where

$$h(\lambda) = \frac{2^{a-b+4n} (\Gamma(\frac{a-b+4n+1}{2}))^2}{\pi \lambda^{a-b+4n}} \mathcal{F}_c(g)(\lambda),$$

and $\mu_{a,b,2n}$ is given by (2.3). Clearly $h \in L^1([0,\infty),\mu_{a,b,2n})$. Thus in view of (5.2) and Theorem 3.2, it is sufficient to prove that $h \in L^2([0,\infty),\mu_{a,b,2n})$. We have

$$\begin{aligned}\int_0^\infty |h(\lambda)|^2 d\mu_{a,b,2n}(\lambda) &= m(a,b,n)\int_0^\infty \lambda^{-(a-b+4n)}|\mathcal{F}_c(g)(\lambda)|^2 d\lambda\\ &= m(a,b,n)\left(\int_0^1+\int_1^\infty\right)\lambda^{-(a-b+4n)}|\mathcal{F}_c(g)(\lambda)|^2 d\lambda\\ &= m(a,b,n)(I_1+I_2),\end{aligned}$$

where $m(a,b,n) = 4^{\frac{a-b+1}{2}}\pi^{-2}(\Gamma(\frac{a-b+4n+1}{2}))^2$. Now by (5.1) there exist a positive constant k such that

$$I_1 \le k\int_0^1 \lambda^{2\eta-(a-b+4n)}d\lambda = \frac{k}{2[\eta-(\frac{a-b-1+4n}{2})]} < \infty.$$

Thus from the Plancherel theorem for the cosine transform, we obtain

$$\begin{aligned} I_2 &= \int_1^\infty \lambda^{-(a-b+4n)}|\mathcal{F}_c(g)(\lambda)|^2 d\lambda\\ &\le \int_0^\infty |\mathcal{F}_c(g)(\lambda)|^2 d\lambda\\ &= \frac{\pi}{2}\int_0^\infty |g(x)|^2dx < \infty.\end{aligned}$$

Thus proof is completed. □

Lemma 5.2. *Let $0 \ne g \in L^1\bigcap L^2([0,\infty),dx)$ such that $\mathcal{F}_c(g) \in L^1([0,\infty),dx)$ and satisfying there exists $\eta > (a-b+4n)$ such that*

$$\mathcal{F}_c(g)(\lambda) = o(\lambda^n) \tag{5.3}$$

as $\lambda \to 0$. Then $\chi g \in L^2_{a,b,n}$ is a generalized wavelet and $\mathcal{F}_\Delta(\chi g) \in L^\infty((0,\infty),dx)$.

Proof. By making use of equation (5.3) and Lemma 5.1, we can easily see that $\chi g \in L^2_{a,b,n}$, $\mathcal{F}_\Delta(\chi g)$ is bounded and

$$\mathcal{F}_\Delta(\chi g)(\lambda) = o(\lambda^{\eta-(a-b+4n)}) \text{ as } \lambda \to 0.$$

Thus by Remark 4.1(i), the function χg satisfies admissibility condition (4.1). Thus proof is completed. □

The classical continuous wavelet transform on $[0,\infty)$ is defined for suitable functions by

$$\mathcal{W}_g(f)(\alpha,\beta) = \frac{1}{\alpha}\int_0^\infty f(x)\overline{\sigma_\beta(g_\alpha)(x)}dx, \tag{5.4}$$

where $\alpha > 0,\ \beta \geq 0$ and $g \in L^\infty((0,\infty), dx)$ is a classical wavelet on $[0,\infty)$ i.e. satisfying the admissibility condition

$$0 < C(g) = \int_0^\infty |\mathcal{F}_c(g)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty \tag{5.5}$$

A more complete and detailed discussion of the properties of the classical continuous wavelet transform on $[0,\infty)$ can be found in [14].

Remark 5.1. *(i) According to [14], each function satisfying the conditions of Lemma 5.2 is a classical wavelet on $[0,\infty)$.*

(ii) In view of (3.12), (4.1) and (5.5), $g \in D(\mathbb{R})$ is a generalized wavelet if and only if ${}^t\chi g$ is a classical wavelet and we have $c({}^t\chi g) = c_g$.

The following statement provides a formula relating the generalized continuous wavelet transform to the classical one.

Lemma 5.3. *Let g be as in Lemma 5.2. Then for all $f \in L^p_{a,b,n},\ p = 1$ or 2, we have*

$$\Phi_{\chi g}(f)(\alpha,\beta) = \frac{1}{\alpha^{a-b+4n}} \chi[\mathcal{W}_g({}^t\chi f)(\alpha,\cdot)](\beta).$$

Proof. We have by (4.6) that

$$\begin{aligned}
\Phi_{\chi g}(f)(\alpha,\beta) &= \frac{1}{\alpha^{a-b+1+4n}} f\#[\chi(\overline{g}_\alpha)](\beta) \\
&= \frac{1}{\alpha^{a-b+1+4n}} \chi[{}^t\chi f \star \overline{g}_\alpha](\beta) \\
&= \frac{1}{\alpha^{a-b+4n}} \chi[\mathcal{W}_g({}^t\chi f)(\alpha,\cdot)](\beta).
\end{aligned}$$

This completes the proof. □

A combination of Theorems 4.2-4.3 with Lemmas 5.2-5.3 yields

Theorem 5.1. *Let g be as in Lemma 5.2. Then we have the following inversion formulas for ${}^t\chi$*

(i) If $f \in L^1_{a,b,n}$ and $\mathcal{F}_\Delta(f) \in L^1_{a,b,2n}$ then for almost all $x \geq 0$ we have

$$f(x) = \frac{1}{C_{\chi g}} \int_0^\infty \left(\int_0^\infty \chi[\mathcal{W}_g({}^t\chi f)(\alpha,\cdot)](\beta)(\chi g)_{\alpha,\beta}(x)\beta^{a-b} d\beta \right) \frac{d\alpha}{\alpha^{a-b+1+4n}}.$$

(ii) For $f \in L^1_{a,b,n} \bigcap L^2_{a,b,n}$ and $0 < \epsilon < \delta < \infty$, the function

$$f^{\epsilon,\delta}(x) = \frac{1}{C_{\chi g}} \int_{\epsilon}^{\delta} \int_{0}^{\infty} \chi[\mathcal{W}_g({}^t\chi f)](\alpha, \cdot)](\beta)(\chi g)_{\alpha,\beta}(x)\beta^{a-b} d\beta \frac{d\alpha}{\alpha^{a-b+1+4n}}$$

satisfies

$$\lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \|f^{\epsilon,\delta} - f\|_{2,a,b,n} = 0.$$

Remark 5.2. *(i) If we set $a = \alpha + \frac{3}{4}$, $b = -\alpha - \frac{1}{4}$ throughout this paper then it reduces to the results studied by R.F. Al Subaie and M.A. Mourou in the paper entitled "The continuous wavelet transform for a Bessel type operator on the half line" published in Mathematics and Statistics 1(4): 196-203, 2013.*

(ii) Author claims that the result obtained in this paper are more general than that of Al Subaie and M.A. Mourou.

References

[1] R.F. Al Subai and M.A. Mourou, Transformation operators associated with a Bessel type operator on the half line and certain of their applications, to appear in Tamsui Oxford Journal of Mathematics.

[2] C.K. Chui, An Introduction to wavelets, Academic press, 1992.

[3] I. Daubechies, Ten lectures on wavelets, CBMS-NSF Regional conference series in Applied Mathematics, Vol.61, SIAM, Philadelphia, 1992.

[4] P. Goupilland, A. Grossmann and J. Morlet, Cycle octave and related transforms in seismic signal analysis Geoexploration 23 (1984), 85-102.

[5] A. Grossmann and J. Morlet, Decomposition of Hardy functions into square integrable wavelets of constant shape, SIAM J. Math. Anal. Vol.15, No.4, (1984), 723-736.

[6] M. Holschneider, Inverse Radon transform through inverse wavelet transform, Inverse Problems 7(1991), 853-861.

[7] M. Holschneider, Wavelets: An Analysis Tool, Clarendon Press, Oxford, 1995.

[8] T.H. Koornwinder, The continuous wavelet transform, Vol.1 Wavelets: An elementary Treatment of Theory and Applications. Edited by T.H. Koornwinder, World Scientific, 1993, 27-48.

[9] Y. Meyer, Wavelets and Operators, Cambridge University Press, Cambridge, 1992.

[10] M.A. Mourru and K. Trimeche, Inversion of the weyl integral transform and the random transform on $\mathbb{R}^n$ using generalized wavelets, Monatshefte fur Mathematik, 126(1998), 73-83.

[11] M.A. Mourru and K. Trimeche, Calderm's formuls associated with a different operator on $(0, \infty)$ and inversion of the generalized Abel transform, Journal of Fourier Analysis and Applications, 4 (1998), 229-245.

[12] M.A. Mourru, Inversion of the dual Dunkl-Sonine integral transform on $\mathbb{R}$ using Dunkl wavelets, SIGMA, 5(2009), 1-12.

[13] K. Trimeche, Generalized wavelets and hyper-graphs, Gordon and Breach Publishing group, 1997.

[14] K. Trimeche, Generalized Harmonic Analysis and Wavelets packets, Gordon and Breach Science publishers, 2001.

