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CONTINUITY OF BESSEL TYPE WAVELET TRANSFORM AND RELATED RESULTS

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Abstract: In this paper, the continuity of the Bessel type wavelet transform B_ψ of the function ϕ in terms of a mother wavelet ψ is investigeted on certain distribution spaces when the Hankel type transform of ψ defined by $\hat{\psi}(x,y) \in C^\infty(\mathbb{R}_+^2)$. Finally a sobolev type space boundedness result is obtained.

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Keywords: Bessel type wavelet transform, Hankel type transform, Bessel type operator, Sobolev type space.

INTRODUCTION

The Hankel type transformation is defined by

$$\hat{\phi}(x) = \left(h_{\alpha,\beta}\phi\right)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\phi(y)dy, x \in (0,\infty) \tag{1}$$

where $J_{\alpha-\beta}(x)$ represents the Bessel type function of the first kind and order $\alpha-\beta$. Throughout this paper we shall assume that $(\alpha-\beta) \geq -1/2$ and $\phi \in L^1(0,\infty)$. The inversion formula for (1) [[2], p.239] is given by

$$\phi(y) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\left(h_{\alpha,\beta}\phi\right)(x)dx, y \in (0,\infty) \tag{2}$$

Zemanian[6] has extended the above transformation to distributions. For every $(\alpha-\beta) \in (0,\infty)$, he introduced the space $H_{\alpha,\beta}(0,\infty)$ consisting of all infinitely differentiable functions ϕ defined on $(0,\infty)$, such that for all $m,k \in \mathbb{N}_0$, the quantities

$$\rho_{m,k}^{\alpha,\beta} = \sup_{x\in(0,\infty)} \left| x^m \left(x^{-1}\frac{d}{dx} \right)^k x^{2\beta-1}\phi(x) \right| < \infty. \tag{3}$$

Using theory of $H_{\alpha,\beta}$ space of Zemanian[6], Pathak and Dixit[3] investigated the Bessel type wavelet transform B_ψ defined as follows:

$$(B_\psi \phi)(b,a) = \int_0^\infty (bu)^{\alpha+\beta} J_{\alpha-\beta}(bu)\hat{\phi}(u)\overline{\hat{\psi}(au)}du \tag{4}$$

where $\hat{\phi}(u) = \left(h_{\alpha,\beta}\phi\right)(u)$.

Let us assume that for every real number λ, $\hat{\psi}$ satisfies

$$(1+x)^l \left| \left(x^{-1}\frac{d}{dx} \right)^a \left(y^{-1}\frac{d}{dx} \right)^b \hat{\psi}(xy) \right| \leq C_{a,b,l}(1+y)^{\lambda-b} \tag{5}$$

for all $a,b,l \in \mathbb{N}_0$, where $C_{a,b,l} > 0$ is a constant and $\hat{\psi}$ denotes the Hankel type transform of the basic wavelet ψ. The class of all such wavelet $\hat{\psi}$ is denoted by H^λ.

This permits us to define the Hankel type transform with respect to x of $\hat{\psi}(ax)$

$$h_{\alpha,\beta}\left[\left(\overline{h_{\alpha,\beta}(\psi)}\right)\right](a\xi) = \int_0^\infty (x\xi)^{\alpha+\beta} J_{\alpha-\beta}(x\xi)\left(\overline{h_{\alpha,\beta}\psi}\right)(ax)dx. \tag{6}$$

We follow the notations and terminology of [2, 4, 5, 7].

The differential operator $\Delta_{\alpha,\beta,x}$ is defined by

$$\begin{aligned}\Delta_{\alpha,\beta,x} &= x^{2\beta-1}D_x x^{4\alpha}D_x x^{2\beta-1}\\ &= (2\beta-1)(4\alpha+2\beta-2)x^{4(\alpha+\beta-1)} + 2(2\alpha+2\beta-1)x^{4\alpha+4\beta-3}D_x\\ &+ x^{2(2\alpha+2\beta-1)}D_x^2.\end{aligned} \tag{7}$$

If we set $\alpha = \frac{1}{4}+\frac{\mu}{2}, \beta = \frac{1}{4}-\frac{\mu}{2}$ in (7), it reduces to $x^{-\mu-\frac{1}{2}}D_x x^{2\mu+1}D_x x^{-\mu-\frac{1}{2}} \equiv S_\mu = D_x^2 + \frac{(1-4\mu^2)}{4x^2}$ which is studied in Zemanian[7].

From [2, 4], we know that for any $\phi \in H_{\alpha,\beta}$,

$$h_{\alpha,\beta}\left(\Delta_{\alpha,\beta}\phi\right) = -y^2 h_{\alpha,\beta}\phi \tag{8}$$

$$\left(x^{-1}\frac{d}{dx}\right)^k(\psi\phi) = \sum_{i=0}^{k}\binom{k}{i}\left(x^{-1}\frac{d}{dx}\right)^i\phi\left(x^{-1}\frac{d}{dx}\right)^{k-i}\psi \tag{9}$$

also from [2], we have

$$\Delta_{\alpha,\beta,x}^r\phi(x) = \sum_{j=0}^{r} b_j x^{2j+2\alpha}\left(x^{-1}\frac{d}{dx}\right)^{r+j}\left(x^{2\beta-1}\phi(x)\right) \tag{10}$$

where b_j are constants depending only on $\alpha-\beta$.

Definition 1.1 *A tempered distribution* $\phi \in H_{\alpha,\beta}'(0,\infty)$ *is said to belong to the Sobolev space* $G^{s,p}_{\alpha,\beta}, s, (\alpha-\beta)\in(0,\infty), 1 \le p < \infty$, *if its Hankel type transform* $h_{\alpha,\beta}\phi$ *coresponds to a locally integrable function over* $(0,\infty)$ *such that*

$$\|\phi\|_{G^{s,p}_{\alpha,\beta}(0,\infty)} = \left(\int_0^\infty \left|\left(1+\xi^2\right)^s\left(h_{\alpha,\beta}\phi\right)(\xi)\right|^p\right)^{1/p} < \infty. \tag{11}$$

THE BESSEL TYPE WAVELET TRANSFORM

Lee[1] has defined the space $B_{\alpha,\beta,b}$ and $\Upsilon^{2q}_{\alpha,\beta,b}$ as follows:

Definition 2.1 *We say that* $\phi \in B_{\alpha,\beta,b}$ *if* ϕ *is smooth function* $\phi(x) = 0$ *for* $x > b$ *and*

$$\rho^{\alpha,\beta}_{b,k}(\phi) = \sup_{x\in(0,\infty)}\left|\left(x^{-1}\frac{d}{dx}\right)^k x^{2\beta-1}\phi(x)\right| < \infty, k = 0,1,2,\ldots \tag{12}$$

where $b > 0$ is a constant and $(\alpha-\beta)$ is a real number.

Definition 2.2 *For each* $q = 1,2,3,\ldots, \Phi \in \Upsilon^{2q}_{\alpha,\beta,b}$ *if* $z^{2\beta-1}\Phi$ *is an even entire function and*

$$\lambda^{\alpha,\beta,2q}_{b,k}(\Phi) = \sup_{z=x+iy}\left|e^{-by^{2q}} z^{2q+2\beta-1}\Phi(z)\right| < \infty, k = 0,1,2,\ldots \tag{13}$$

where $\Phi = \left(h_{\alpha,\beta}\phi\right), b > 0$ is constant and $(\alpha-\beta)$ is a real number.

The topology of the spaces $B_{\alpha,\beta,b}$ and $\Upsilon^{2q}_{\alpha,\beta,b}$ are generated by the seminorms $\{\rho^{\alpha,\beta}_{b,k}\}_{k=0}^\infty$ and $\{\lambda^{\alpha,\beta,2q}_{b,k}\}_{k=0}^\infty$ respectively. From Definitions (2.1) and (2.2), it follows that $B_{\alpha,\beta,b}$ and $\Upsilon^{2q}_{\alpha,\beta,b}$ are Frechet spaces. We define

$$\sigma_{b,k}^{\alpha,\beta}(\phi)=\max_{0\le i\le k}\rho_{b,i}^{\alpha,\beta} \tag{14}$$

$$w_{b,k}^{\alpha,\beta,2q}(\phi)=\max_{0\le i\le k}\lambda_{b,i}^{\alpha,\beta,2q}(\phi). \tag{15}$$

Then $\sigma_{b,k}^{\alpha,\beta}(\phi)$ and $w_{b,k}^{\alpha,\beta,2q}(\phi)$ define a norm on the spaces $B_{\alpha,\beta,b}$ and $\Upsilon_{\alpha,\beta,b}^{2q}$ respectively. Following techniques of Zemanian[7], we can write

$$x^i\left(x^{-1}\frac{d}{dx}\right)^n x^{2\beta-1}h_{\alpha,\beta}\phi(x)=\int_0^\infty y^{2(\alpha-\beta)+2n+i+1}\left(y^{-1}\frac{d}{dy}\right)^i\left(y^{2\beta-1}\phi(y)\right)(xy)^{-(\alpha-\beta+n)}$$
$$\times J_{\alpha-\beta+i+n}(xy)dy. \tag{16}$$

Theorem 2.1 *The Bessel type wavelet transform* B_ψ *is a continuous linear mapping of* $B_{\alpha,\beta,b}$ *into* $\Upsilon_{\alpha,\beta,b}^{2q}$.

Proof. Let $z=x+iy$ and $(\alpha-\beta)\ge -1/2$, the Bessel type wavelet transform B_ψ has the representation

$$(B_\psi\phi)(z,a)=\int_0^b (zu)^{\alpha+\beta}J_{\alpha-\beta}(zu)(h_{\alpha,\beta}\phi)(u)\overline{(h_{\alpha,\beta}\psi)(au)}du$$

$(\alpha-\beta)\ge -1/2$, with $b>0$ and $(h_{\alpha,\beta}\phi)(u)\overline{(h_{\alpha,\beta}\psi)(au)}\in L^2(0,b)$ if and only if $(B_\psi\phi)(z,a)\in L^2(0,\infty), z^{2\beta-1}(B_\psi\phi)(z,a)$ is an even entire function of z and there exists a constant C such that

$$\left|(B_\psi\phi)(z,a)\right|\le Ce^{b/y},\ for\ all\ z.$$

If $\phi\in B_{\alpha,\beta,b}$, then

$$(B_\psi\phi)(z,a)=\int_0^b (zu)^{\alpha+\beta}J_{\alpha-\beta}(zu)(h_{\alpha,\beta}\phi)(u)\overline{(h_{\alpha,\beta}\psi)(au)}du$$
$$=h_{\alpha,\beta}\left[(h_{\alpha,\beta}\phi)(u)\overline{(h_{\alpha,\beta}\psi)(au)}\right](z).$$

Applying the technique of the Zemanian for fixed a, from (16),

$$z^{2k+2\beta-1}(B_\psi\phi)(z,a)=\int_0^b u^{2k+4\alpha}\left[(u^{-1}D)^{2k}u^{2\beta-1}\overline{(h_{\alpha,\beta}\psi)(au)}(h_{\alpha,\beta}\phi)(u)\right]$$
$$\times\left[(zu)^{-(\alpha-\beta)}J_{\alpha-\beta+2k}(zu)\right]du.$$

So that

$$\left|e^{-by^{2q}}z^{2k+2\beta-1}(B_\psi\phi)(z,a)\right|\le\int_0^b\left|u^{2k+4\alpha}\left[(u^{-1}D)^{2k}u^{2\beta-1}\overline{(h_{\alpha,\beta}\psi)(au)}(h_{\alpha,\beta}\phi)(u)\right]\right.$$
$$\times\left.\left[(zu)^{-(\alpha-\beta)}J_{\alpha-\beta+2k}(zu)\right]e^{-by^{2q}}\right|du.$$
$$\le\int_0^b\left|u^{2k+4\alpha}\sum_{s=0}^{2k}\binom{2k}{s}(u^{-1}D)^s\overline{(h_{\alpha,\beta}\psi)(au)}(u^{-1}D)^{2k-s}\right|$$
$$\times\left|u^{2\beta-1}(h_{\alpha,\beta}\phi)(u)\right|\sup_{z,u}\left|\left[(zu)^{-(\alpha-\beta)}J_{\alpha-\beta+2k}(zu)\right]e^{-by^{2q}}\right|du$$
$$\le\int_0^b\sum_{s=0}^{2k}\binom{2k}{s}\sup_u\left|(1+u)^{2k+4\alpha}(u^{-1}D)^s\overline{\hat\psi(au)}\right|$$
$$\times\sup_u\left|(u^{-1}D)^{2k-s}u^{2\beta-1}(h_{\alpha,\beta}\phi)(u)\right|$$

$$\times \sup_{z,u}\left|\left[(zu)^{-(\alpha-\beta)} J_{\alpha-\beta+2k}(zu)\right]e^{-by^{2q}}\right|du$$

Applying inequalities (5) and (12), then from the above, we have

$$\int_0^b \sum_{s=0}^{2k}\binom{2k}{s}(1+u)^{2k+4\alpha} C_s (1+u)^w \rho_{b,2k-s}^{\alpha,\beta}\left(h_{\alpha,\beta}\phi\right)\sup_{z,u}\left|\left[(zu)^{-(\alpha-\beta)} J_{\alpha-\beta+2k}(zu)\right]e^{-by^{2q}}\right|du$$

$$\leq \sum_{s=0}^{2k}\binom{2k}{s} C_s \rho_{b,2k-s}^{\alpha,\beta}\left(h_{\alpha,\beta}\phi\right)\sup_{z,u}\left|\left[(zu)^{-(\alpha-\beta)} J_{\alpha-\beta+2k}(zu)\right]e^{-by^{2k}}\right|\int_0^b (1+u)^{2k+2(\alpha-\beta)+w+1}du.$$

We note that for all z such that $|z|\leq 1$

$$\left|z^{-(\alpha-\beta)} J_{\alpha-\beta+2k}(z)\right| \leq \frac{2^{-(\alpha-\beta)}e}{(\alpha-\beta)!}$$

and for $|z|>1$

$$\left|z^{-(\alpha-\beta)} J_{\alpha-\beta+2k}(z)\right| \leq C(\pi/2)^{-1}|z|^{2\beta-1} e^{|Imz|}.$$

As $|y|\leq|y|^{2q}$, if $|y|\geq 1$ and $|y|\geq|y|^{2q}$, if $|y|<1$, then

$$\left|(zu)^{-(\alpha-\beta)} J_{\alpha-\beta+2k}(zu)e^{-by^{2q}}\right| \leq \frac{2^{-(\alpha-\beta)}e}{(\alpha-\beta)!}e^{-by^{2q}}, |y|\leq 1$$

$$\leq C(\pi/2)^{-1}|z|^{2\beta-1} e^{-b\left(|y|^{2q}-|y|\right)}, |y|>1.$$

Thus

$$\left|(zu)^{-(\alpha-\beta)} J_{\alpha-\beta+2k}(zu)e^{-by^{2q}}\right| \leq L_1,\ for\ 2\alpha\geq 0.$$

Therefore

$$\left|e^{-by^{2q}} z^{2k+2\beta-1}\left(B_\psi\phi\right)(z,a)\right| \leq L_1\sum_{s=0}^{2k}\binom{2k}{s} C_s \rho_{b,2k-s}^{\alpha,\beta}\left(h_{\alpha,\beta}\phi\right).$$

THE SOBOLEV TYPE SPACE

The Sobolev type space $G_{\alpha,\beta}^{s,p}$ is defined by an equation (11). In the following, we shall make use of the following norm on $G_{\alpha,\beta}^{s,p}\left((0,\infty)\times(0,\infty)\right)$ in the proof of the boundedness result

$$\|\phi\|_{G_{\alpha,\beta}^{s,p}((0,\infty)\times(0,\infty))} = \left(\int_0^\infty\int_0^\infty\left|(1+\xi^2)^s(1+\eta^2)^s\overline{\left(h_{\alpha,\beta}\phi\right)(\xi,\eta)}\right|^p d\xi d\eta\right)^{1/p}$$

$\phi\in H_{\alpha,\beta}'\left((0,\infty)\times(0,\infty)\right)$.

Lemma 3.1 *Let us assume that for any positive real number* w, $\hat{\psi}(x)$ *satisfies*

$$\left|\left(x^{-1}\frac{d}{dx}\right)^l x^{2\beta-1}\hat{\psi}(x)\right| \leq C_{l,w}(1+x)^{w-l} \tag{17}$$

for all $l\in \mathrm{N}_0$, then there exists a positive constant C' such that

$$\left|h_{\alpha,\beta}\left[\overline{\left(h_{\alpha,\beta}\psi\right)}\right](a\xi)\right| \leq C'(1+a)^{w+2l+2\alpha}(1+\xi^2)^{-l}. \tag{18}$$

Proof. From the Definition (6), we know that

$$h_{\alpha,\beta}\left[\overline{\left(h_{\alpha,\beta}\psi\right)}\right](a\xi) = \int_0^\infty (x\xi)^{\alpha+\beta} J_{\alpha-\beta}(x\xi)\left(h_{\alpha,\beta}\psi\right)(ax)dx.$$

So that from [4],

$$(1+\xi)^2 h_{\alpha,\beta}\left[\overline{(h_{\alpha,\beta}\psi)}\right](a\xi) = \int_0^\infty (x\xi)^{\alpha+\beta} J_{\alpha-\beta}(x\xi)\left(1-\Delta_{\alpha,\beta,x}\right)^l \left(h_{\alpha,\beta}\psi\right)(ax)dx \qquad (19)$$

for all $\xi, a \in (0,\infty), l \in \mathrm{N}_0$ and $\Delta_{\alpha,\beta,x}$ defined as (7).

Now

$$\left(1-\Delta_{\alpha,\beta,x}\right)^l \left(h_{\alpha,\beta}\psi\right)(ax) = \sum_{r=0}^{l}\binom{l}{r}(-1)^r \Delta_{\alpha,\beta,x}^r \left(h_{\alpha,\beta}\psi\right)(ax)$$

$$= \sum_{r=0}^{l}\binom{l}{r}(-1)^r \sum_{j=0}^{r} b_j x^{2j+2\alpha}\left(x^{-1}\frac{d}{dx}\right)^{r+j}$$

$$\times\left(x^{2\beta-1}\left(h_{\alpha,\beta}\psi\right)(ax)\right) \qquad (20)$$

Hence by (19) and (20), and inequality (17), we have

$$\left|h_{\alpha,\beta}\left[\overline{(h_{\alpha,\beta}\psi)}\right](a\xi)\right| \le Q_{\alpha,\beta}(1+a)^{w+2l+2\alpha}(1+\xi^2)^{-l}\sum_{r=0}^{l}\sum_{j=0}^{r} b_j C_{r+j,w}\binom{l}{r}$$

$$\times \int_0^\infty (1+x)^{2j+2(\alpha-\beta)+w+1-2l}dx$$

Choosing $l - j > (\alpha-\beta) + \frac{w}{2} + 1$, we conclude that

$$\left|h_{\alpha,\beta}\left[\overline{(h_{\alpha,\beta}\psi)}\right](a\xi)\right| \le C'(1+a)^{w+2l+2\alpha}(1+\xi^2)^{-l}$$

Definition 3.1 *Let* $\left(h_{\alpha,\beta}\psi\right)(a\xi)$ *be a wavelet in* H^w *defined by (5), then the Bessel type wavelet transform* B_ψ *defined by*

$$\left(B_\psi\phi\right)(y,x) = \int_0^\infty (y\eta)^{\alpha+\beta} J_{\alpha-\beta}(y\eta)\overline{\left(h_{\alpha,\beta}\psi\right)(x\eta)}\left(h_{\alpha,\beta}\phi\right)(\eta)d\eta \qquad (21)$$

exists for $\phi \in H_{\alpha,\beta}(0,\infty)$.

Theorem 3.1 *For any wavelet* $h_{\alpha,\beta}\psi \in H^w$*, the Bessel type wavelet transform* $\left(B_\psi\phi\right)(y,x)$ *admits the representation*

$$\left(B_\psi\phi\right)(y,x) = \int_0^\infty\int_0^\infty (x\xi)^{\alpha+\beta} J_{\alpha-\beta}(x\xi)(y\eta)^{\alpha+\beta} J_{\alpha-\beta}(y\eta)\overline{\left(h_{\alpha,\beta}\psi\right)(\xi\eta)}\left(h_{\alpha,\beta}\phi\right)(\eta)d\xi d\eta \qquad (22)$$

where $\phi \in H_{\alpha,\beta}(0,\infty)$.

Proof. From definition (3.1)

$$\left(B_\psi\phi\right)(y,x) = \int_0^\infty (y\eta)^{\alpha+\beta} J_{\alpha-\beta}(y\eta)\overline{\left(h_{\alpha,\beta}\psi\right)(x\eta)}\left(h_{\alpha,\beta}\phi\right)(\eta)d\eta$$

$$= \int_0^\infty (y\eta)^{\alpha+\beta} J_{\alpha-\beta}(y\eta)\left[\int_0^\infty (x\xi)^{\alpha+\beta} J_{\alpha-\beta}(x\xi)\overline{\left(\hat{h}_{\alpha,\beta}\psi\right)(\xi\eta)}d\xi\right]\left(h_{\alpha,\beta}\phi\right)(\eta)d\eta$$

$$= \int_0^\infty\int_0^\infty (x\xi)^{\alpha+\beta} J_{\alpha-\beta}(x\xi)(y\eta)^{\alpha+\beta} J_{\alpha-\beta}(y\eta)\overline{\left(\hat{h}_{\alpha,\beta}\psi\right)(\xi\eta)}\left(h_{\alpha,\beta}\phi\right)(\eta)d\xi d\eta$$

The last integral exists because $\left(h_{\alpha,\beta}\phi\right)(\eta) \in H_{\alpha,\beta}(0,\infty)$ and $\overline{\left(\hat{h}_{\alpha,\beta}\psi\right)(\xi\eta)}$ satisfies the inequality (18).

Corollary 3.1 *For any basic wavelet* $h_{\alpha,\beta}\psi \in H^w$*, the Bessel type wavelet transform* $h_{\alpha,\beta}\left[B_\psi\phi\right](\xi,\eta)$ *admits the representation*

$$h_{\alpha,\beta}\left[B_\psi\phi\right](\xi,\eta) = \overline{\left(\hat{h}_{\alpha,\beta}\psi\right)(\xi\eta)}\left(\hat{h}_{\alpha,\beta}\phi\right)(\eta) \qquad (23)$$

where $\phi \in H_{\alpha,\beta}(0,\infty)$.

Proof. The right hand side of (23)

$$\overline{(\hat{h}_{\alpha,\beta}\psi)(\xi\eta)}(\hat{h}_{\alpha,\beta}\phi)(\eta) = h_{\alpha,\beta}\left[\overline{(h_{\alpha,\beta}\psi)(\xi\eta)}\right]h_{\alpha,\beta}\left[(h_{\alpha,\beta}\phi)(\eta)\right]$$
$$= \left[\int_0^\infty (x\xi)^{\alpha+\beta} J_{\alpha-\beta}(x\xi)\overline{(h_{\alpha,\beta}\psi)(x\eta)}dx\right]$$
$$\times\left[\int_0^\infty (y\eta)^{\alpha+\beta} J_{\alpha-\beta}(y\eta)(h_{\alpha,\beta}\phi)(y)dy\right]$$
$$= \int_0^\infty\int_0^\infty (x\xi)^{\alpha+\beta} J_{\alpha-\beta}(x\xi)(y\eta)^{\alpha+\beta} J_{\alpha-\beta}(y\eta)\overline{(h_{\alpha,\beta}\psi)(x\eta)}$$
$$\times (h_{\alpha,\beta}\phi)(y)dxdy$$
$$= h_{\alpha,\beta}\left[B_\psi\phi\right](\xi,\eta).$$

Theorem 3.2 *Let* $h_{\alpha,\beta}\psi \in H^w$ *and* $(B_\psi\phi)(x,y)$ *be the Bessel type wavelet transform then there exists* $D>0$ *such that for* $w\in(0,\infty)$ *and* $l\in \mathrm{N}_0$,

$$\| (B_\psi\phi)\|_{G^{s,p}_{\alpha,\beta}((0,\infty)\times(0,\infty))} \le D\,\| (h_{\alpha,\beta}\phi)\|_{G^{s+(w+2l+2\alpha)/2,p}_{\alpha,\beta}(0,\infty)}$$

for all $\phi \in H_{\alpha,\beta}(0,\infty)$.

Proof. Using Lemma (3.1), we have

$$\Big)^{1/p}.$$

We note that

$$(1+\eta)^{w+2l+2\alpha} \le 2^{(w+2l+2\alpha)/2}(1+\eta^2)^{(w+2l+2\alpha)/2}, w\ge 0$$

and

$$(1+\eta)^{w+2l+2\alpha} \le (1+\eta^2)^{(w+2l+2\alpha)/2}, w<0.$$

Therefore

$$(1+\eta)^{w+2l+2\alpha} \le \max\left(1,2^{(w+2l+2\alpha)/2}\right)(1+\eta^2)^{(w+2l+2\alpha)/2}.$$

Hence

$$\| (B_\psi\phi)\|_{G^{s,p}_{\alpha,\beta}((0,\infty)\times(0,\infty))} \le C'\left(\int_0^\infty\int_0^\infty \left|(1+\xi^2)^{s-l}\max\left(1,2^{(w+2l+2\alpha)/2}\right)(1+\eta^2)^{s+(w+2l+2\alpha)/2}\left(\hat{h}_{\alpha,\beta}\phi\right)(\eta)\right|^p d\xi d\eta\right)^{1/p} \le C''\left(\int_0^\infty |(1+\right.$$

where C'' is certain constant. The ξ integral is convergent as l can be chosen large enough so that

$$\| (B_\psi\phi)\|_{G^{s,p}_{\alpha,\beta}((0,\infty)\times(0,\infty))} \le D\,\| (h_{\alpha,\beta}\phi)\|_{G^{s+(w+2l+2\alpha)/2,p}_{\alpha,\beta}(0,\infty)}.$$

PRODUCT OF BESSEL TYPE WAVELET TRANSFORMS

Let B_{ψ_1} and B_{ψ_2} be two Bessel type wavelet transforms of $\phi \in H_{\alpha,\beta}(0,\infty)$ defined as

$$\left(B_{\psi_1}\phi\right)(b,a) = B_1(b,a) = \int_0^\infty (bu)^{\alpha+\beta} J_{\alpha-\beta}(bu)\overline{\hat{\psi}_1(au)}\hat{\phi}(u)du, \quad (24)$$

and

$$\left(B_{\psi_2}\phi\right)(d,c) = B_2(d,c) = \int_0^\infty (du)^{\alpha+\beta} J_{\alpha-\beta}(du)\overline{\hat{\psi}_2(cu)}\hat{\phi}(u)du. \tag{25}$$

Then their product $B_{\psi_1} \circ B_{\psi_2}$ (or $B_1 \circ B_2$) is defined by

$$B(b,a,c) = (B_1 \circ B_2)(b,a,c)$$

$$= \int_0^\infty (bu)^{\alpha+\beta} J_{\alpha-\beta}(bu)\overline{\hat{\psi}_1(au)} h_{\alpha,\beta}\left[B_{\psi_2}\phi\right](u,c)du \tag{26}$$

$$= \int_0^\infty (bu)^{\alpha+\beta} J_{\alpha-\beta}(bu)\overline{\hat{\psi}_1(au)}\,\overline{\hat{\psi}_2(cu)}\hat{\phi}(u)du \tag{27}$$

$$= \int_0^\infty (bu)^{\alpha+\beta} J_{\alpha-\beta}(bu)\chi(a,c,u)\hat{\phi}(u)du,$$

where $\hat{\phi}$ denotes the Hankel type transformation of ϕ and $\chi(a,c,u) = \overline{\hat{\psi}_1(au)}\,\overline{\hat{\psi}_2(cu)}$, provided the integral is convergent.

Theorem 4.1 *Let* $\overline{\hat{\psi}_1(au)} \in H^{w_1}$ *and* $\overline{\hat{\psi}_2(au)} \in H^{w_2}$, *then for constant* $D > 0$ *and* $w_1, w_2 \in (0,\infty)$,

$$\| \left(B_{\psi_1} B_{\psi_2}\phi\right)(b,a,c) \|_{G^{s,p}_{\alpha,\beta}((0,\infty)\times(0,\infty))} \le D \| \hat{\phi} \|_{G^{s+(w_1+w_2)/2,p}_{\alpha,\beta}(0,\infty)}.$$

Proof. By definition (26), we have

$$\left(B_{\psi_1} B_{\psi_2}\phi\right)(b,a,c) = \int_0^\infty (bu)^{\alpha+\beta} J_{\alpha-\beta}(bu)\overline{\hat{\psi}_1(au)} h_{\alpha,\beta}\left[B_{\psi_2}\phi\right](u,c)du.$$

From (27), it follows that $\left(B_{\psi_1} B_{\psi_2}\phi\right)$ has Hankel type transform equal to $\left[\overline{\hat{\psi}_1(au)}\,\overline{\hat{\psi}_2(au)}\hat{\phi}(u)\right]$.

Therefore

$$\| \left(B_{\psi_1} B_{\psi_2}\phi\right)(b,a,c) \|_{G^{s,p}_{\alpha,\beta}((0,\infty)\times(0,\infty))} = \left(\int_0^\infty \int_0^\infty \left| (1+u^2)^s (1+a^2)^s \overline{\hat{\psi}_1(au)}\,\overline{\hat{\psi}_2(au)}\hat{\phi}(u)du \right|^p dadu \right)^{1/p}.$$

Since from (5)

$$|\hat{\psi}_1(au)| \le C_{w_1,l}(1+a)^{-l}(1+u)^{w_1}$$

$$\le C_{w_1,l} \max(1,2^{-l/2})(1+a^2)^{-l/2} \max(1,2^{w_1/2})(1+u^2)^{w_1/2}$$

and

$$|\hat{\psi}_2(au)| \le C_{w_2,l}(1+a)^{-l}(1+u)^{w_2}$$

$$\le C_{w_2,l} \max(1,2^{-l/2})(1+a^2)^{-l/2} \max(1,2^{w_2/2})(1+u^2)^{w_2/2}.$$

Therefore

$$\| \left(B_{\psi_1} B_{\psi_2}\phi\right) \|_{G^{s,p}_{\alpha,\beta}((0,\infty)\times(0,\infty))} \le \left(\int_0^\infty \int_0^\infty \left| (1+u^2)^s (1+a^2)^s C_{w_1,l} C_{w_2,l}{}' (1+u^2)^{(w_1+w_2)/2} (1+a^2)^{-l} \hat{\phi}(u) \right|^p dadu \right)^{1/p} \le C_{w_1,w_2,l}\Big(\int$$

where $C_{w_1,w_2,l}$ is certain positive constant.

The $a-$ integral can be made convergent by choosing l sufficiently large, so that

$$\| \left(B_{\psi_1} B_{\psi_2}\phi\right) \|_{G^{s,p}_{\alpha,\beta}((0,\infty)\times(0,\infty))} \le D \| \hat{\phi} \|_{G^{s+(w_1+w_2)/2,p}_{\alpha,\beta}(0,\infty)},$$

where D is positive constant.

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